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J. M. T. THOMPSON

BASIC CONCEPTS OF ELASTIC STABILITY

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BASIC CONCEPTS OF ELASTIC STABILITY

by

J. M. T. Thompson*

SUDAER No. 146

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*Visiting Research Associate
and
Research Fellow from Peterhouse, Cambridge

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SUMMARY

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The buckling and snapping behaviour of an elastic structure subjected to a single generalized load is discussed, the analysis being based on a more general study of elastic stability recently presented by the author. The discussion is orientated to high-light the more physical and intuitive aspects of elastic stability, and examples of the various phenomena are indicated. The examples include problems of shell buckling that are of considerable interest at the present time.

The two extreme cases of dead and rigid loading are readily studied with the aid of the general theory. Thus if the 'loading parameter' of the general theory is equated to the magnitude of the load, a direct treatment of the dead-load problem is obtained: in this application the auxiliary loading parameter of the general theory can be identified as the corresponding displacement of the generalized load. Conversely, the loading parameter can be equated to the magnitude of the corresponding displacement, to give a direct treatment of rigid loading: and the auxiliary loading parameter can then be identified (with a change of sign) as the magnitude of the generalized constraining load.

The two direct treatments are entirely distinct when applied to the same structural system. For this reason the problem of rigid loading is finally re-studied in the context of the dead-load formulation. In this manner the inter-relationships between the "external stability" of a structure under dead load, and the "internal stability" of the structure under rigid load can be observed.

The salient analytical features of the paper are summarized in a résumé, and the behaviour of an illustrative buckling model is analysed in an appendix.

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I. INTRODUCTION

The stability of elastic structures has attracted considerable attention in recent years, due to the extensive use of thin metal shells as load-carrying components in missile structures. These thin shells are liable to fail by instability, long before the onset of appreciable material yielding. The buckling behaviour of shell structures is moreover extremely complex, and research workers in this field have been forced to examine the fundamental concepts of elastic stability with great care.

In predicting the instability of a structure or structural component it can frequently be assumed that the structure is subjected to a single generalized conservative load, the magnitude of which increases slowly with time. A theoretical prediction of elastic instability can then be based on a statical analysis of the structure under the given loading system. The success of this statical approach clearly rests on the ability of the analyst to identify, in the equilibrium paths of the structure, the equilibrium state at which the initial stability of the structure will be lost.

If the initial stable equilibrium path of the structure yields a locally maximum value of the load, it is apparent that a further increase in load will give rise to a dynamic snap of the structure. The structure is said to <u>snap</u> from such an extremum. Moreover, with the exclusion of a special case, Poincaré (1885) showed that in the absence of such an extremum the stability of the initial equilibrium path can only be lost at a "point of bifurcation", at which the path intersects a second distinct and continuous path. The structure is said to <u>buckle</u> at such a point of intersection, whether or not a dynamic snap of the structure is involved.

In the light of these observations it is clear that the two phenomena of snapping and buckling are of particular interest to the analyst in his theoretical study of elastic instability.

A theoretical study of this nature will in general be supplemented by the experimental study of a 'model' structure under the same idealized loading system. In such a study it is frequently advantageous to test the model in a 'rigid' loading device, which prescribes values of the corresponding deflection, rather than the magnitude of the generalized load. This <u>rigid</u> loading of the model is in contrast to the approximately <u>dead</u> conditions of prescribed load which will usually be encountered by the structure in service. Thus when such a rigid test is performed, the behaviour of the structure under dead loading must subsequently be infered from the observed behaviour of the model under rigid loading.

Clearly the snapping and buckling behaviour of a structure under a single generalized load, which might be either dead or rigid in nature, is of considerable practical importance. It is the aim of the present paper to throw as much light as possible on this behaviour.

The paper, being thus a discussion of the stability of a 'general' elastic structure, represents a development of the so-called "general theory of elastic stability", which is itself a division of classical mechanics.

Studies in the general theory of elastic stability can be made in terms of generalized coordinates, or in the context of continuum elasticity. The former approach, being the simpler mathematically, is useful for developing new physical concepts, which can later be discussed more rigorously in the context of continuum elasticity. General studies of elastic stability can secondly be loosely classified as either 'linear' or 'non-linear'.

The 'linear' studies are primarily concerned with the critical equilibrium states themselves, and are not concerned with the precise equilibrium path configurations in which these states might be found. Notable 'linear' studies have been made by Westergaard (1922), Ziegler (1956), Pearson (1956), Hill (1957) and Koiter (1962). Of these, Westergaard and Ziegler worked in generalized coordinates and discussed only linear eigenvalue problems, while Pearson, Hill and Koiter studied the more general problem in the context of continuum elasticity.

'Non-linear' studies are concerned with the precise equilibrium path configurations that might be found in the vincinity of a critical equilibrium state. These configurations are of particular interest to the analyst, and form the subject of the present paper. 'Non-linear' studies have been made by Poincaré (1885), Koiter (1945) and Thompson (1963). Koiter worked in the context of continuum elasticity and limited his attention to branching conditions, while Poincaré and Thompson discussed both branching and snapping configurations in terms of generalized coordinates.

The phenomena of snapping and buckling were thus first discussed in the classic paper of Poincaré (1885), which laid the foundations of the general theory of elastic stability. They were subsequently analysed in detail by the author in 1963. This recent study, which demonstrated for the first time the essential inter-relationships between the two phenomena, forms the basis of the present paper.

More technical discussions of structural stability have been presented by Timoshenko (1936) Pfluger (1950), Hoff (1954), Thompson (1961a) Ashwell (1962) and Libove (1962).

In the present paper the results of the recent general study of elastic stability (Thompson - 1963) are first summarized in a manner that highlights the physical aspects of the various phenomena. The theory is then applied to the specific problem of an elastic structure loaded by a single generalized conservative load.

It is seen that the 'loading parameter' of the general theory can be equated to either the magnitude of the generalized load, or to the magnitude of the corresponding deflection. In this manner the general theory yields a direct treatment of the two extreme cases of dead and rigid loading, which in the terminology of Ashwell (1962) are associated with the "external" and "internal" stability of the structure respectively.

The two direct treatments are however entirely distinct, and the problem of rigid loading is finally discussed in the context of the dead-load analysis. In this manner the inter-relationships between the concepts of internal and external stability are examined.

Following the general study, the paper is limited in scope to a consideration of the most general snapping and buckling configurations that can arise in the equilibrium paths of a structure. Within this limitation the paper represents an exposition of the basic concepts of elastic stability, that is broad and intuitive rather than essentially rigorous in nature.

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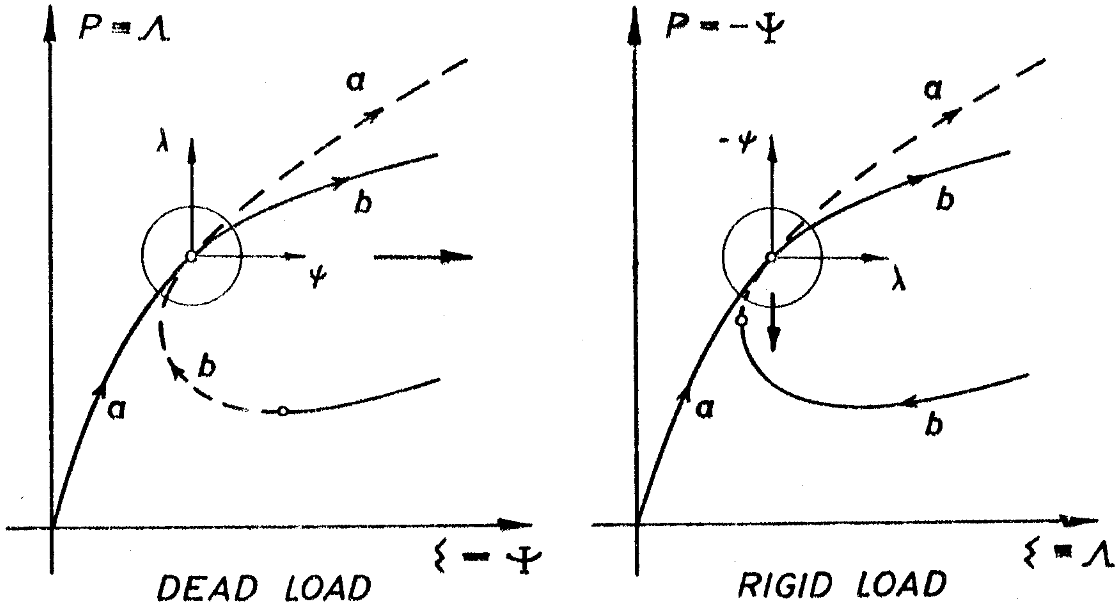
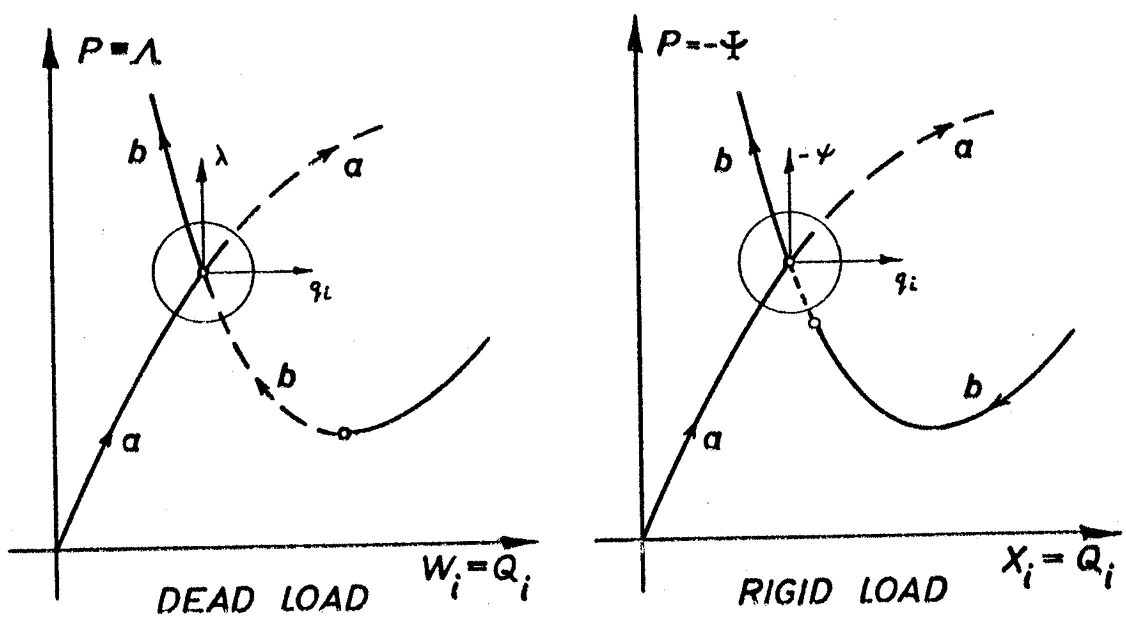


a) LOAD AGAINST THE CORRESPONDING DEFLECTION



b) LOAD AGAINST A GENERAL LATERAL DEFLECTION

FIG. 1. A GENERAL BUCKLING CONFIGURATION

II. GENERAL REMARKS

It has been indicated that an elastic structure, which might be subjected to either dead or rigid loading, will in general lose its initial stability by either buckling or snapping. The structure is said to buckle when the initial stable equilibrium path loses its stability at a point of intersection (bifurcation). The structure is said to snap when the path loses its stability on yielding the first locally maximum value of the loading parameter.

The buckling phenomenon is illustrated by the equilibrium paths of Figure 1, in which P is the magnitude of a generalized load, and ϵ is the corresponding deflection. These paths, if the initial path were linear, could represent the behaviour of an Euler strut constrained laterally by a non-linear spring (Tsien 1942); they could also represent the behaviour of certain rigid-jointed triangular frames (Britvec 1960).

In this and in subsequent figures a stable path is indicated by a continuous curve, and an unstable path by a broken curve. A dynamic snap of the structure is represented by a heavy arrow.

The point of intersection reproduced in Figure 1 is the most general that can arise (Thompson 1963), and is exhibited by structures that encounter different conditions as they deflect in either of the two possible directions. Thus in the problem of the constrained Euler strut, the spring is assumed to be 'hard' in compression, (that is to say the incremental stiffness increases with deflection) and 'soft' in tension. The strut has thus a definite preference to deflect in the 'soft' direction, and the two branches of the post-buckling path are distinct on a plot of the load against the end-shortening.

Most structures are however designed with a high degree of symmetry, and this general buckling behaviour is fairly rare. Thus the majority of structures encounter identical conditions as they deflect in either of the two possible directions, and the two branches of the post-buckling curve are consequently coincident on a plot of the load against the corresponding deflection. This non-general behaviour is illustrated

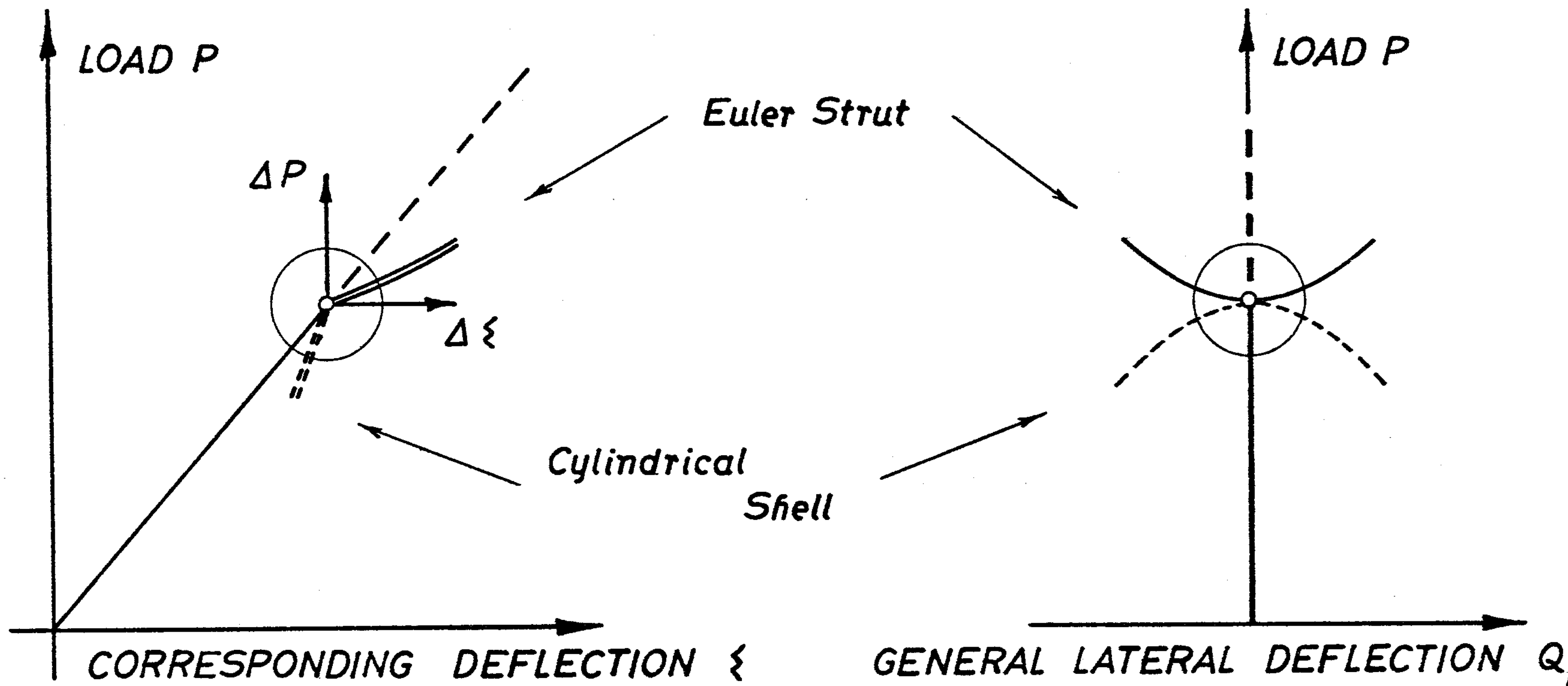


FIG. 2. TWO NON-GENERAL BUCKLING CONFIGURATIONS

by the unconstrained Euler strut, for which the two branches rise and are stable. A further example is provided by the axially-loaded cylindrical panel (Koiter 1955), for which the two branches fall and are consequently unstable. These two non-general buckling configurations are illustrated in Figure 2, in which the indicated regions of stability and instability are applicable both to dead and rigid loading conditions.

Following the recently developed general analysis (Thompson 1963), attention is restricted in the present paper to the general buckling condition of Figure 1. Clearly, however it is most desirable to extend the general analysis to include the special cases of Figure 2 in the near future.

The snapping phenomenon is illustrated by the equilibrium paths of Figure 3. These paths might represent the behaviour of the constrained Euler strut in the presence of an initial imperfection (Tsien 1942): they might also represent the behaviour of an initially imperfect cylindrical shell under axial compression (Donnell and Wan 1950).

A perfectly general extremum in the equilibrium paths of a structure will define a perfectly general snapping configuration, and attention is restricted in the paper to such a configuration. It is clear that the majority of snapping configurations encountered in practice will be perfectly general in this sense.

The present paper is designed to throw as much light as possible on these two phenomena of snapping and buckling, under both dead and rigid loading conditions. The discussion is easily extended, following the lines of Thompson (1961 a), to the problem of semi-rigid loading which prevails in many experimental analyses.

The behaviour of a simple two-degree-of-freedom buckling model is analysed in the Appendix and is used to illustrate the salient features of the paper.

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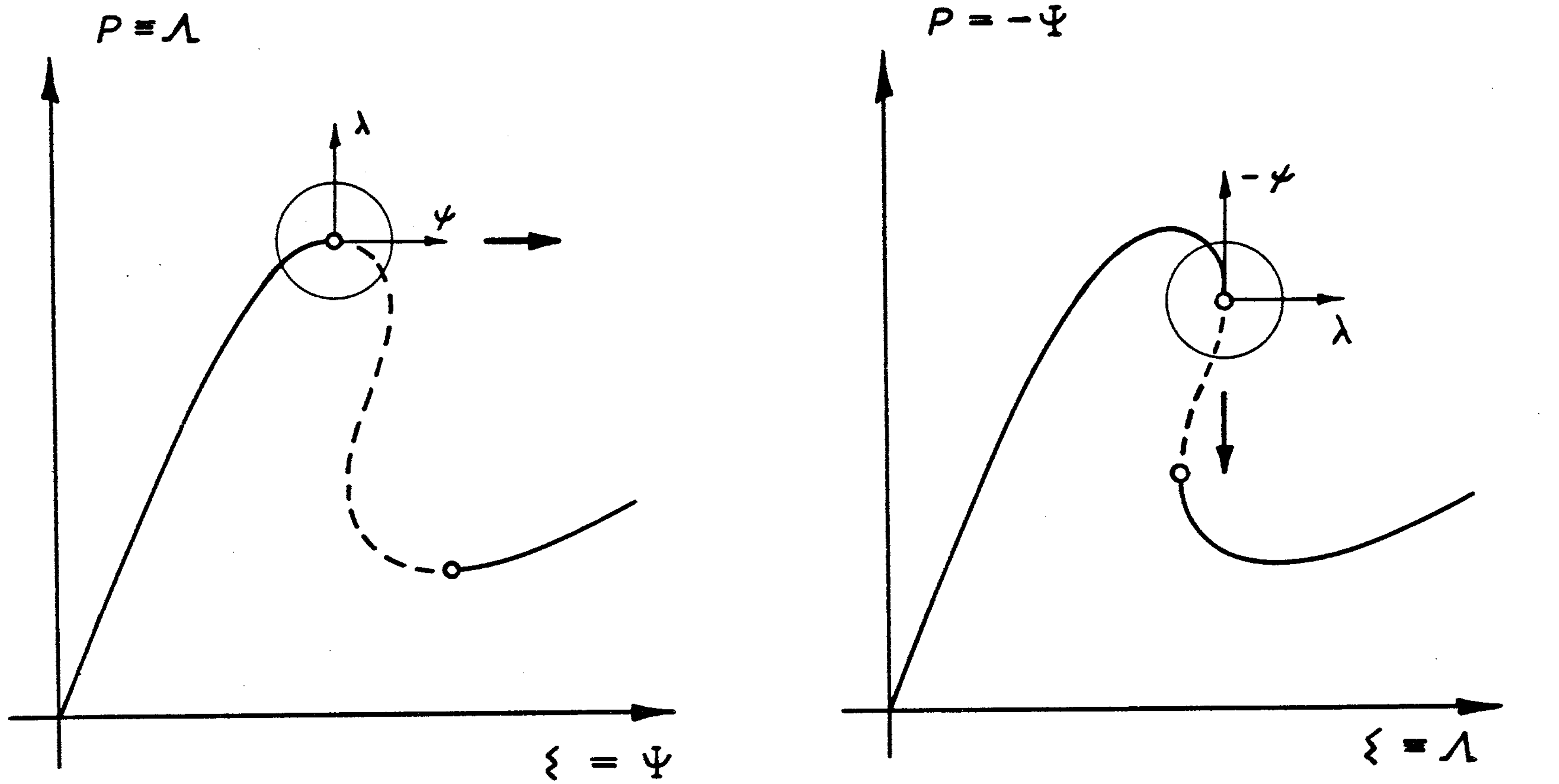


a) DEAD LOAD b) RIGID LOAD

FIG. 3. SNAPPING CONFIGURATIONS

III. FORMULATION OF THE PROBLEM

The problem that we wish to study is that of a conservative elastic structure loaded by a single generalized conservative load (Figure 4). A more precise definition of the structural system under consideration can be formulated from the energy function introduced in section 5.

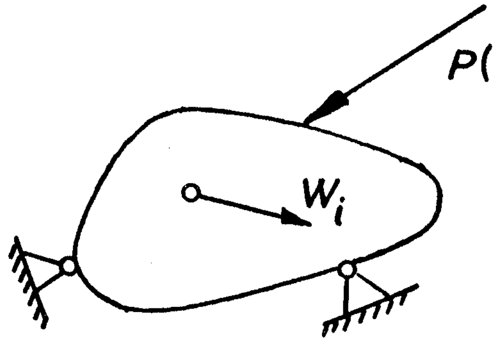
In particular we wish to discuss the stability of the structure under both dead and rigid loading conditions.

In the case of dead loading, the magnitude, P, of the generalized load is imposed on the structure at a given state of loading; and the slow variation of P with time (t) describes the loading process. In the contrasting case of rigid loading, the magnitude, ϵ, of the corresponding deflection is prescribed as a function of time.

It is assumed that the deformations of the structure can be analyzed into mode-forms, the amplitudes of which will supply a set of generalized coordinates for the structure. It is further assumed that the behaviour of the structure can be described satisfactorily by the use of a large but finite number of coordinates.

In a direct treatment of dead loading, it is convenient to introduce a set of m generalized coordinates, W_i, which defines the deformed state of the structure when the corresponding deflection, ϵ, is free to vary (Figure 4a). In this case, ϵ, and the strain energy of the structure, U, are both single-valued functions of the W_i. Following Ashwell (1962), we shall introduce the term <u>external stability</u> to describe the stability of the structure under dead loading.

For a direct treatment of rigid loading, it is convenient to introduce a set of $m - 1$ generalized coordinates, X_i, which defines the state of the structure under an imposed value of ϵ (Figure 4b). In this case the single-valued strain energy function can be written as $U(X_i, \epsilon)$. We shall use the term <u>internal stability</u> to describe the stability of the structure under rigid loading.



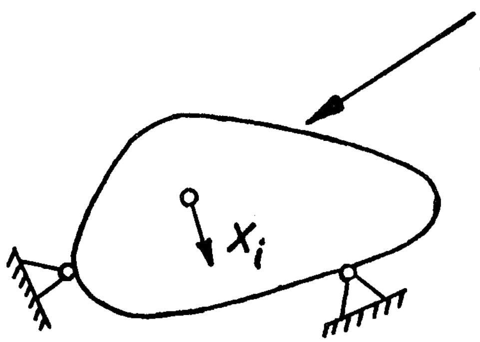
$$P(t)\,,\quad \xi(W_i) = -\frac{\partial V}{\partial P}$$

$$i = 1 \text{ to } m$$

$$U = U(W_i)$$

$$V = U(W_i) - P\,\xi(W_i)$$

a) DEAD LOADING



$$\xi(t)\,,\quad P(X_i,\xi) = \frac{\partial V}{\partial \xi}$$

$$i = 1 \text{ to } m-1$$

$$U = U(X_i\,,\,\xi)$$

$$V = U(X_i\,,\,\xi)$$

b) RIGID LOADING

FIG. 4. STRUCTURAL SYSTEM

IV. GENERAL THEORY

4.1. Introduction. The general study of elastic stability recently presented by the author is summarized in the present section.

The analysis of this general study is essentially non-linear, in the sense that the critical path configurations are discussed analytically. In the present summary, however, the analysis has been linearized (in an incremental sense), so the critical path configurations of snapping and buckling are only discussed in a qualitative manner.

The presentation of the theory is moreover essentially new. The more physical aspects of the theory have been emphasized, and the resulting treatment provides a useful insight into the snapping and buckling phenomena. The introduction of vector notation has facilitated the presentation.

The general theory is applicable to a structural system described by an energy function $V(Q_i, \Lambda)$, where the Q_i are n generalized coordinates, and Λ is a loading parameter. Thus the theory will yield a direct treatment of dead loading (section 5.1) if we set $\Lambda \equiv P$, or a direct treatment of rigid loading (section 5.2) if we set $\Lambda \equiv \epsilon$.

Moreover we shall introduce in the general theory an auxiliary loading parameter Ψ, where $\Psi \equiv -(\partial V/\partial \Lambda)$. Then, when we set $\Lambda \equiv P$ in the treatment of dead loading, Ψ can be identified as the corresponding deflection ϵ. Similarly when Λ is set equal to ϵ, the auxiliary parameter (with a change of sign) can be identified as the magnitude of the load, so that $\Psi = -P$

4.2. Structural System. Let us consider the behaviour of a conservative structure described by n generalized coordinates Q_i.

We shall define a coordinate space by associating the Q_i with rectangular axes in n-space, and we shall introduce the unit vectors $\vec{h}_i$ in the Q_i directions. It is however perhaps worth noting that a more elegant and intrinsic treatment is possible if it is assumed that a metric coordinate space is specified à priori.

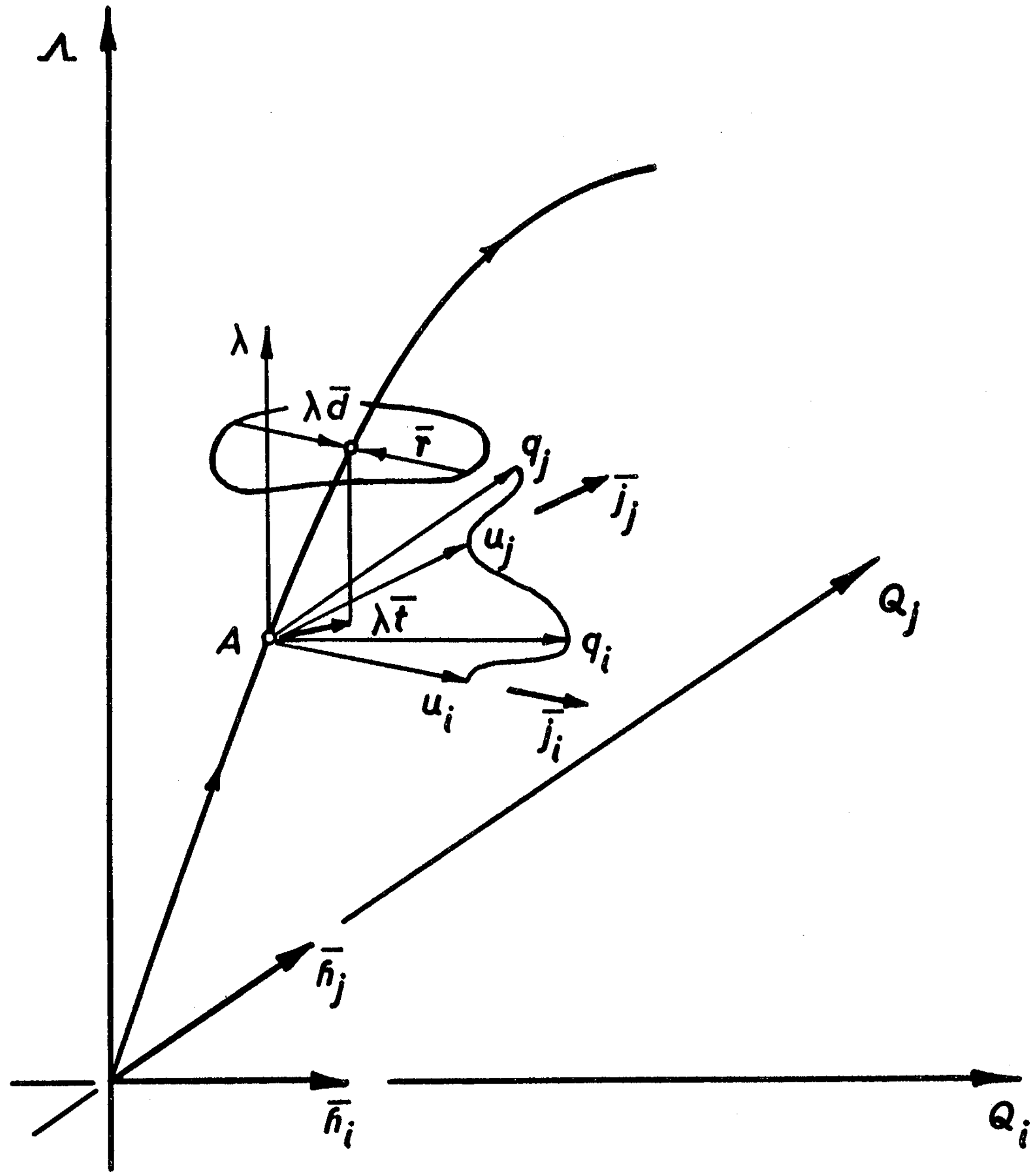


FIG. 5. GENERALIZED LOAD-COORDINATE SPACE

Let us now introduce a loading parameter Λ, such that at different but constant values of Λ, the total potential energy of the system can be written as $V(Q_i, \Lambda)$. The single-valued energy function, V, is assumed to be continuous and well-behaved.

We now associate Λ with an axis orthogonal to all the $\vec{h}_i$, to define an $(n+1)$-dimensional load-coordinate space. For convenience we shall introduce no vector in the Λ-direction, so that all vectors will lie in the n-dimensional coordinate space.

The n equilibrium equations $(\partial V/\partial Q_i) = 0$ define a series of paths in the load-coordinate space as Λ is varied. A single path is shown in the schematic diagram of Figure 5.

These n equilibrium equations can be represented by the single vector equation $\overrightarrow{\text{grad } V} = 0$. Here, in agreement with our convention that vectors (and consequently vector equations) refer only to the coordinate space, $\overrightarrow{\text{grad } V}$ is the gradient of V at a constant value of Λ.

4.3. <u>Principal Coordinates and Stability Coefficients.</u> Let us consider the equilibrium state A of Figure 5, and introduce the lower case symbols, λ, q_i, etc., to denote changes in the variables from this state. At state A all the first derivatives, $\partial V/\partial Q_i$, are zero, and to a first approximation we can write the change of V in the vicinity of state A as

$$v = \frac{1}{2} \sum \sum \frac{\partial^2 V}{\partial Q_i \partial Q_j} q_i q_j + \left\{ \frac{\partial V}{\partial \Lambda} + \sum \frac{\partial^2 V}{\partial \Lambda \partial Q_i} q_i \right\} \lambda + \frac{1}{2} \frac{\partial^2 V}{\partial \Lambda^2} \lambda^2 \tag{1}$$

Here, and subsequently, all summations range from 1 to n. The points at which derivatives are to be evaluated will always be apparent from the context: the derivatives are here to be evaluated at the basic state A.

The stability of state A is determined by the second variation of V at constant Λ,

$$\delta^2 V \equiv \left| V \right|_{\lambda=0} = \frac{1}{2} \sum \sum \frac{\partial^2 V}{\partial Q_i \partial Q_j} \; q_i q_j$$

It is convenient to reduce this expression to a sum of squares by a suitable change of coordinates. We shall thus introduce a new set of coordinates, u_i, by means of the linear transformation represented by the n equations

$$q_r = \sum \alpha_{ri} u_i \quad \text{for all } r$$

Here the determinant $|\alpha_{ij}|$ is non-zero, and the α_{ij} coefficients are chosen to eliminate the cross-terms of $\delta^2 V$.

Then in the <u>principal coordinates</u>, u_i, we have

$$\underline{\delta^2 V = \frac{1}{2} \sum \frac{\partial^2 V}{\partial u_i^2} u_i^2} \qquad \text{———————} \qquad (2)$$

and the stability of the basic state is determined by the set of n "<u>stability coefficients</u>", $C_i \equiv (\partial^2 V / \partial u_i^2)$, assuming that these are all non-zero.

The reduction of a quadratic form to a sum of squares can be accomplished in an infinite number of ways, so the stability coefficients as defined above are not unique. To eliminate this undesired lack of uniqueness we shall specify that the u_i axes shall be rectangular in coordinate space, and to the same scale as the q_i axes. The stability coefficients and the associated principal axes are now uniquely defined for a given system and a given basic state. (To be precise we should add "for a given coordinate space". The coordinate space has been defined arbitrarily in the present paper; as previously noted however, a more elegant treatment is possible if it is assumed that a metric coordinate space is given à priori.)

We shall introduce a set of unit vectors $\vec{j}_i$ in the u_i directions, noting that there is only an arbitrary distinction between the $+u_i$ and

$-u_i$ directions. The transformation of coordinates is shown schematically in Figure 5.

4.4. Equilibrium Paths. Equilibrium paths in the vicinity of state A, assuming this to be a general non-critical equilibrium state, are defined to a first approximation by the n scalar equations $(\partial v/\partial q_i) = 0$, where v is given by equation (1). These equations can be replaced by the single vector equation

$$\underline{\Delta\, \overrightarrow{\text{grad } V} = \lambda \overrightarrow{\frac{\partial}{\partial \Lambda} \text{grad } V} + \overrightarrow{\sum q_i \frac{\partial}{\partial Q_i} \text{grad } V}} \qquad (3)$$

$$\underline{= 0}$$

The first vector of the right-hand side can be interpreted, with a change of sign, as a small disturbing force $\lambda \vec{d}$ (in n-space) associated with a small increase in Λ. Thus under an imposed increase of Λ (represented by λ), the system will move to a new configuration in which this disturbing force is balanced by the restraining force

$$\vec{r} = -\overrightarrow{\sum q_i \frac{\partial}{\partial Q_i} \text{grad } V}$$

We denote the small movement of the system by the vector $\lambda\vec{t}$, so that $\vec{t}$ is the response of the system to a unit change in Λ.

The above discussion is represented schematically in Figure 5.

The disturbing force produced by a unit change in Λ is given by

$$\vec{d} = -\overrightarrow{\frac{\partial}{\partial \Lambda} \text{grad } V}$$

which can be written as

$$\overrightarrow{\text{grad}\left(-\frac{\partial V}{\partial \Lambda}\right)}$$

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Thus the scalar variable

$$\Psi \equiv -\frac{\partial V}{\partial \Lambda}$$

is clearly of particular importance in representing the 'action' of the loading parameter Λ. We shall refer to Ψ as the <u>auxiliary loading parameter</u>.

Introducing the coefficients

$$S_r \equiv \frac{\partial \Psi}{\partial u_r} = -\frac{\partial^2 V}{\partial \Lambda \partial u_r} \qquad \text{and} \qquad T \equiv \frac{\partial \Psi}{\partial \Lambda}$$

we can write to a first approximation

$$\psi \equiv \Delta \Psi = \sum S_i u_i + T\lambda$$

and

$$\vec{d} = \overrightarrow{\mathrm{grad}\ \Psi} = \sum S_i \vec{j}_i \qquad (4)$$

The unique equilibrium path in the vicinity of a general (non-critical) basic state is readily located in the principal coordinate system. Thus rewriting equation (1) in the principal coordinates u_i we have

$$v = \frac{1}{2} \sum \frac{\partial^2 V}{\partial u_i^2} u_i^2 + \left\{ \frac{\partial V}{\partial \Lambda} + \sum \frac{\partial^2 V}{\partial \Lambda \partial u_i} u_i \right\} \lambda + \frac{1}{2} \frac{\partial^2 V}{\partial \Lambda^2} \lambda^2 \qquad (5)$$

and setting

$$\frac{\partial v}{\partial u_r} = 0$$

gives

$$u_r = -\left(\frac{\partial^2 V}{\partial \Lambda \partial u_r} \bigg/ \frac{\partial^2 V}{\partial u_r^2} \right) \lambda \, ;$$

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that is

$$u_r = \lambda(S_r/C_r) \text{ for all } r \qquad (6)$$

Thus the incremental response of the system is given by

$$\vec{t} = \sum (S_i/C_i)\vec{j_i} \qquad (7)$$

Moreover the change in Ψ along the path is given by $\psi = [\Sigma(S_i^2/C_i) + T]\lambda$; so that we can write

$$\left|\frac{\partial\Psi}{\partial\Lambda}\right|_{path} = \sum\left(\frac{S_i^2}{C_i}\right) + T \qquad (8)$$

If all the stability (incremental stiffness) coefficients at state A are equal, the response vector $\vec{t}$ (equation 7) will have the same direction as the disturbance vector $\vec{d}$ (equation 4). When the coefficients are unequal, the response vector will have increased components in the directions of reduced stiffness.

4.5. Critical Equilibrium State. As the basic state under consideration is allowed to move along an equilibrium path, the u_i axis will rotate, and the coefficients S_i and C_i will vary in a continuous manner.

Thus as the system is loaded from an initial stable state, the initial stability can only be lost at a critical equilibrium state for which at least one of the stability coefficients (C_1 say) is zero. The other stability coefficients will in general be positive and non-zero at the critical equilibrium state.

Since

$$\left|\frac{\partial^2 V}{\partial Q_i \partial Q_j}\right| \cdot \left|\alpha_{ij}\right|^2 = \left|\frac{\partial^2 V}{\partial u_i \partial u_j}\right|$$

$$= C_1 \cdot C_2 \cdot C_3 \cdots\cdots C_n$$

we can observe that the two-determinants

$$\left|\frac{\partial^2 V}{\partial Q_i \partial Q_j}\right| \qquad \text{and} \qquad \left|\frac{\partial^2 V}{\partial u_i \partial u_j}\right|$$

will be zero at a critical equilibrium state.

With $C_1 = 0$ and $C_s > 0$ for $s \neq 1$, the second variation of V, $\delta^2 V$, is zero in the two directions $\pm\vec{j_1}$, and positive in all other directions. We observe moreover that the restraining force $\vec{r}$ can now have no component in the $\pm\vec{j_1}$ directions; that is to say $\vec{r}\cdot\vec{j_1} = 0$.

4.6. Snapping Condition. In general the disturbance vector $\vec{d}$ will have a finite component in the direction $\vec{j_1}$. That is to say

$$\left.\begin{array}{l} \vec{d}\cdot\vec{j_1} \neq 0 \\ \\ \text{or alternatively} \\ \\ S_1 \neq 0 \end{array}\right\} \tag{9}$$

We shall observe that under this condition the loss of stability is in general associated with a snapping point.

We assume without loss of generality that $\vec{d}$ has a positive component in the direction $+\vec{j_1}$, so that S_1 is positive.

Let us allow the basic state A to move along a stable equilibrium path with increasing Λ towards a critical equilibrium state for which $C_1 = 0$ (and $C_s > 0$ for $s \neq 1$). Thus the stability coefficient C_1 will be dropping to zero.

Since the disturbance vector $\vec{d}$ has a component in the direction of decreasing stiffness, the component of the response vector in this direction will be increasing (equation 7). Finally, at the critical equilibrium state, the response vector $\vec{t}$ will have the direction of $+\vec{j_1}$, and an infinite magnitude.

In this manner the equilibrium path will in general reach a maximum value of Λ at the critical equilibrium state, as shown in Figure 6.

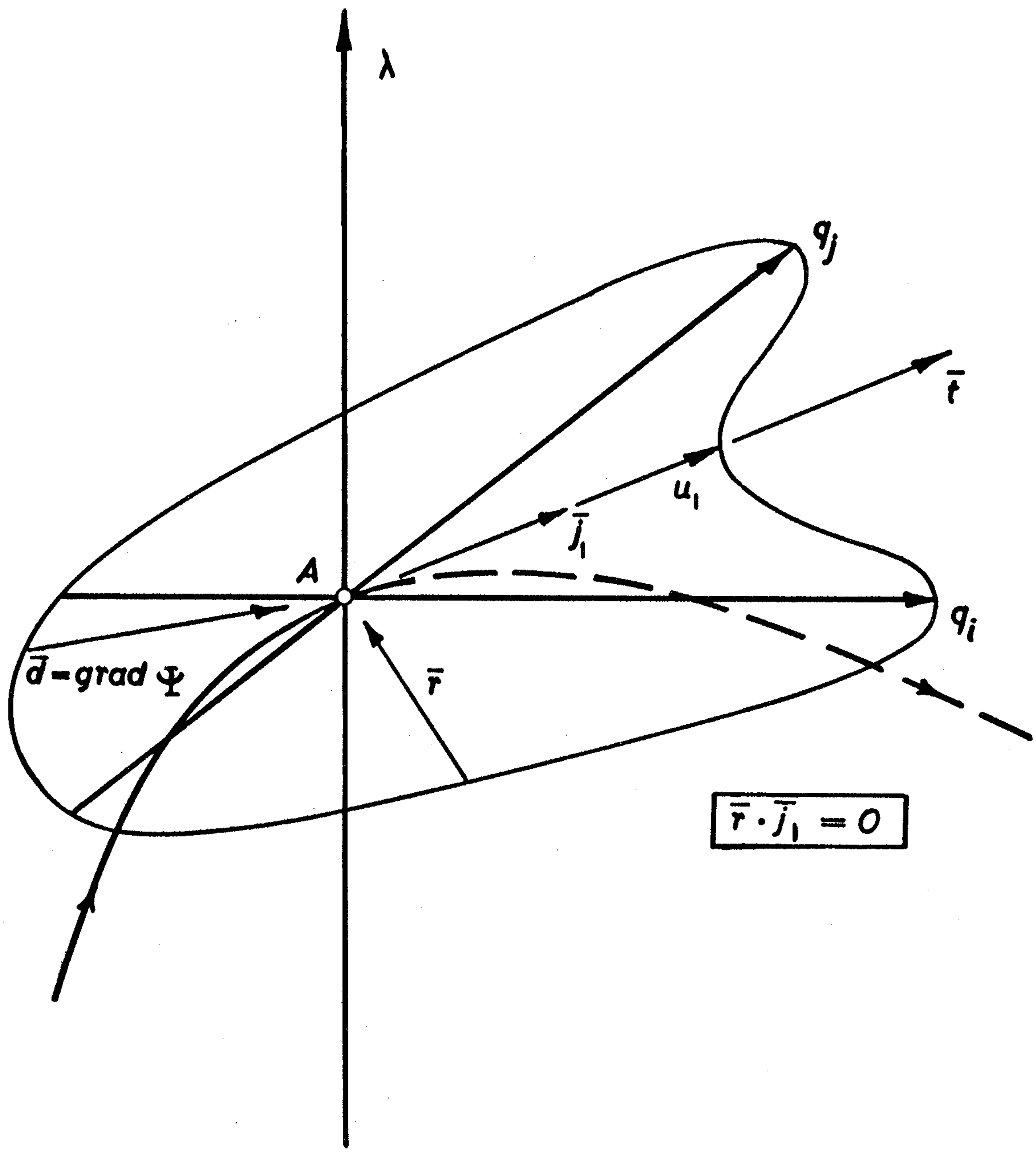


FIG. 6. SNAPPING CONFIGURATION IN GENERALIZED LOAD-COORDINATE SPACE ($\bar{d} \cdot \bar{j}_1 \neq 0$)

This maximum can be interpreted as the inability of the restraining force $\vec{r}$ to provide a (positive) component in the direction $(-)\vec{j_1}$: it should be remembered that we are dealing with the case in which the disturbance vector $\vec{d}$ has a finite (positive) component in the direction $(+)\vec{j_1}$.

Moving along the equilibrium path through the critical equilibrium state, C_1 drops to zero and becomes negative. Thus beyond the maximum the path is unstable (with respect to u_1), as indicated in Figure 6 by the broken curve. The system loses its stability at this maximum, and will snap dynamically to a new equilibrium configuration.

The path defines a smooth maximum on a plot of Λ against any Q_i, or on a plot of Λ against Ψ.

4.7. <u>Buckling Condition</u>. A special case of considerable interest arises when, at the critical equilibrium state, the disturbance vector $\vec{d}$ has no component in the direction of zero stiffness. That is when

$$\left.\begin{array}{l} \vec{d} \cdot \vec{j_1} = 0 \\ \text{or alternatively} \\ S_1 = 0 \end{array}\right\} \qquad \text{———} \qquad (10)$$

We shall observe that under this condition the loss of stability is in general associated with a point of intersection (bifurcation).

Qualitatively it is clear that, the disturbance vector having no component in the direction of zero stiffness, the infinite response will no longer arise. In other words, while the restraining force is still unable to provide a component in the $\vec{j_1}$ direction, the disturbing force now has no component in this direction: ignoring questions of stability, the structure can still support an increase in load.

Thus as we move along a stable path toward a critical equilibrium state with C_1 and S_1 dropping to zero, we see the component of the response vector in the direction of incipient instability, t_1, tending to an indeterminate form. That is, from equation (7) we have

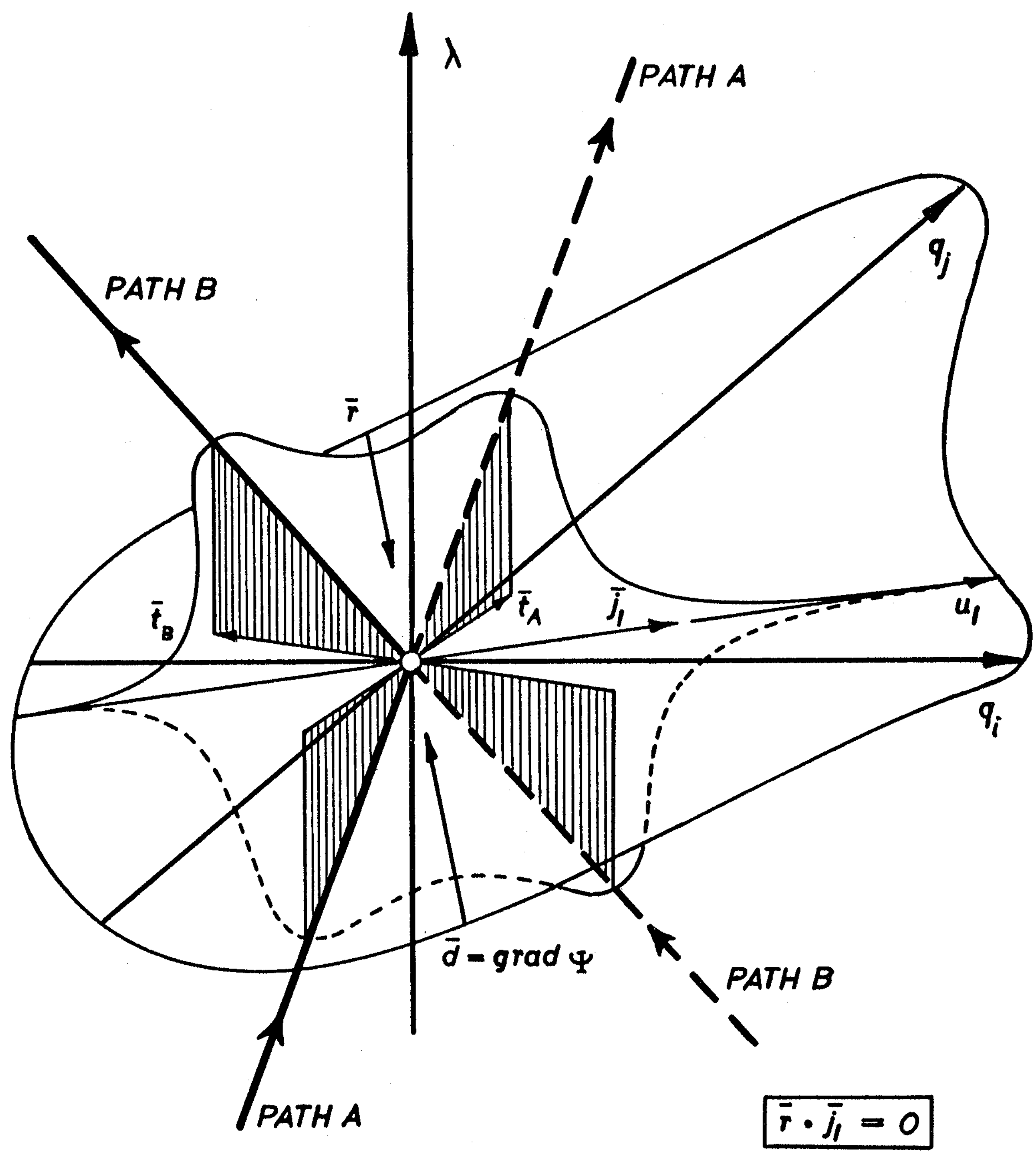


FIG. 7. GENERAL BUCKLING CONFIGURATION IN GENERALIZED LOAD-COORDINATE SPACE $(\bar{d} \cdot \bar{j}_1 = 0)$

$$t_1 = \frac{S_1}{C_1} \rightarrow \frac{0}{0}$$

A large-deflection non-linear analysis (Thompson 1963) shows that this component has in general two alternative finite values at the critical equilibrium state. Thus the simultaneous vanishing of S_1 and C_1 yields a point of intersection (bifurcation) at which two distinct and continuous equilibrium paths intersect (Figure 7).

Moving along the initially stable path (A) with increasing Λ, the stability coefficient C_1 drops to zero at the critical equilibrium state, and then becomes negative. Conversely the second path (B) is initially unstable, so that the stability coefficient corresponding to C_1 is originally negative: with increasing Λ this stability coefficient increases to zero at the critical equilibrium state, and then becomes positive. Thus in the terminology of Poincaré (1885) there is an "exchange of stabilities" between the two paths.

Although the second path is stable above the branching point, the critical equilibrium state itself is unstable with respect to u_1, so that in the presence of the smallest disturbance, the system will lose its stability at this state and snap dynamically to a new equilibrium configuration.

The two paths intersect in an arbitrary manner on a plot of Λ against any Q_i. However since there is no indeterminacy in the non-critical components of the response vector [i.e. $|t_r|_A = |t_r|_B = (S_r/C_r)$ for $r \neq 1$] and since the disturbance vector $\vec{d} = \overrightarrow{\text{grad } \Psi}$ has no component in the direction $\vec{j}_1$, it follows that the two curves intersect tangentially on a plot of Λ against Ψ. The common slope is moreover given by equation (8) if we set

$$\frac{S_1^2}{C_1} = \frac{0^2}{0} = 0$$

These details of the intersection can be seen on Figure 7.

V. APPLICATION OF THE GENERAL THEORY

5.1. External Stability of a Structure. It is clear that the preceding general analysis is directly applicable to the problem of a structure under a single generalized dead load which has been formulated in section 3. Thus if we merely set $\Lambda \equiv P$, and $Q_i \equiv W_i$ (so that $n = m$), the general discussion could be reproduced as it stands to supply a direct treatment of the dead-load problem.

In this direct treatment the energy function V, now a function of P and the W_i, can be written as

$$V = U(W_i) - P\epsilon(W_i) \tag{11}$$

where $U(W_i)$ is the strain energy of the structure.

We can observe that V is now linear in $\Lambda(\equiv P)$, which was not necessarily the case in the general theory. Moreover since

$$\frac{\partial V}{\partial P} = - \epsilon(W_i)$$

we have the significant result that when the loading parameter Λ is equated to the magnitude of the generalized load P, the auxiliary loading parameter Ψ will represent the corresponding deflection of the load, ϵ.

The equilibrium equations of the system can be written as

$$\left|\frac{\partial V}{\partial W_i}\right|_{\text{const } P} = \frac{\partial U}{\partial W_i} - P\,\frac{\partial \epsilon}{\partial W_i} = 0 \tag{12}$$

The results of the general analysis can now be interpreted as the well-known conclusions for dead loading, which are summarized as follows.

Under a slowly increasing load the external stability of a structure will be 'lost' at the first critical equilibrium state encountered. This critical equilibrium state will in general correspond to a snapping point; under certain conditions it will correspond to a point of intersection.

At a snapping point the equilibrium path of the structure reaches the first locally maximum value of the load, as shown in Figure 3a. The path in general yields a smooth maximum on a plot of the load against its corresponding deflection, or on a plot of the load against any 'general lateral deflection'. (In the present discussion the term 'general lateral deflection' is used to describe any general coordinate, as distinct from any special coordinate such as the corresponding deflection or one of the principal coordinates.) After the maximum the path is unstable.

At a general point of intersection the initially stable path intersects a second distinct and continuous path (Figure 1a). The critical equilibrium state itself is unstable, and the paths exhibit an exchange of stabilities with increasing load. The form of the intersection is arbitrary on a plot of the load against any 'general lateral deflection', but the two paths intersect tangentially on a plot of the load against its corresponding deflection.

It should be observed that both the snapping condition, and the general buckling condition give rise to a dynamic snap of the structure.

One further result, applicable exclusively to the dead-load problem, can finally be indicated. Since V is now linear in $\Lambda(\equiv P)$, the coefficient $T \equiv -(\partial^2 V/\partial \Lambda^2)$ is identically zero, so that equation (8) becomes

$$\underline{\left|\frac{\partial \Psi}{\partial \Lambda}\right|_{\text{Path}} \equiv \left|\frac{\partial \epsilon}{\partial P}\right|_{\text{Path}} = \sum \frac{S_i^2}{C_i}} \qquad (13)$$

Thus if $|\partial P/\partial \epsilon|_{\text{Path}}$ is negative, it follows that at least one of the C_i coefficients must be negative. This is a restricted proof of the well-known result that an equilibrium path is externally unstable over any region for which it has a negative slope on a plot of the load against the corresponding deflection.

5.2. Internal Stability of a Structure. The general theory of section 4 is likewise immediately applicable to the rigid-load formulation of section 3 , if we set $\Lambda \equiv \epsilon$ and $Q_i \equiv X_i$. It should be noted that n is now equal to $m - 1$.

The total potential energy function V is now the strain energy function $U(X_i, \epsilon)$, and equilibrium states are defined by the $m - 1$ equations

$$\left|\frac{\partial U}{\partial X_i}\right|_{\text{const } \epsilon} = 0$$

The equilibrium states themselves are of course the same as those defined in the dead-load formulation by the m equations

$$\left|\frac{\partial V}{\partial W_i}\right|_{\text{const } P} = 0$$

Any deformed state of the structure compatible with an imposed value of ϵ, can be held in 'equilibrium' by the application of suitable constraining forces on the X_i coordinates. Then for any such deformed 'equilibrium' state (which might of course be a true equilibrium state for which all the contraining forces are zero) the principle of virtual work gives us the equation

$$\left|\frac{\partial U}{\partial \epsilon}\right|_{\text{const } X_i} = P$$

so that we have the equality $\Psi = - P$. Thus when the loading parameter Λ is equated to the corresponding deflection, ϵ, of a generalized load, the auxiliary loading parameter Ψ will represent, with a change of sign, the magnitude of the constraining load, P.

The results of the general analysis can now be interpreted as follows.

Under a slowly increasing displacement (ϵ) the internal stability of a structure will be 'lost' at the first critical equilibrium state encountered. This critical equilibrium state will in general correspond to a snapping point; under certain conditions it will correspond to a point of intersection.

At a snapping point the equilibrium path of the structure reaches the first locally maximum value of the displacement (ϵ) as shown in Figure 3b. The path in general yields a smooth maximum (of the imposed displacement ϵ) on a plot of the constraining load against the imposed displacement, and on a plot of a 'general lateral deflection' against the imposed displacement. It is perhaps worth noting however that the path is in general incrementally linear with no stationary point on a plot of the constraining load against a 'general lateral deflection'. The path loses its internal stability at the critical equilibrium state, and the system will snap dynamically from this state.

The complete analogy between the dead and rigid-load snapping points is apparent, although for a given structure (Figure 3) the two points are of course essentially unrelated.

At a general point of intersection the initially stable path intersects a second distinct and continuous path. The critical equilibrium state is itself internally unstable, so the system will snap dynamically from this state, although the paths exhibit an exchange of internal stabilities with increasing ϵ.

The general point is characterized by an arbitrary intersection on a plot of the constraining load against a 'general lateral deflection' and by a tangential intersection on a plot of the constraining load against the imposed displacement. The details of this path intersection are thus identical with those observed in the dead-load discussion of the previous section.

Thus there is clearly no distinction between the path intersection observed in the dead-load analysis and that observed in the present rigid-load analysis. In other words the simultaneous vanishing of S_1 and C_1 in the direct treatment of dead loading, describes the same deformed state of a structure as the simultaneous vanishing of S_1 and C_1 in the direct treatment of rigid loading.

Thus a point of intersection in the equilibrium paths of a structure is associated with a 'loss' of both internal and external stability. A structure losing its initial external stability at a general point of intersection will also lose its initial internal stability, as indicated in Figure 1.

VI. INDIRECT ANALYSIS OF INTERNAL STABILITY

6.1. Introduction. The direct treatment of internal stability, while indicating the behaviour of a structure under an imposed displacement, is essentially distinct from the direct treatment of the same structure under an imposed load. Consequently it is instructive to study the problem of internal stability in the context of the dead-load analysis: that is in the context of the general analysis with $\Lambda \equiv P$, and $Q_i \equiv W_i$.

Such a study is made in the present section. The analysis is essentially 'linear' in nature, and the range of validity of the analysis is not explored.

6.2. Structural System. Let us consider the structure of Figure 4, described by the m generalized coordinates W_i and the strain energy function $U(W_i)$, loaded now by the slow variation of the imposed displacement $\epsilon(W_i)$.

Equilibrium states are defined by the equation

$$|\delta U|_{\mathrm{const}\ \epsilon} = 0$$

which can be converted by the introduction of a Lagrange multiplier, ρ, to the m equations

$$\underline{\delta\overline{U} = \frac{\partial U}{\partial W_i} - \rho \frac{\partial \epsilon}{\partial W_i} = 0} \qquad \text{———————} \qquad (14)$$

It can be seen that these equations define (as they must) the same equilibrium states as those defined in the dead-load analysis by equations (12). Moreover the value obtained for the Lagrange multiplier will be the magnitude of the constraining load, P.

6.3. Stability Coefficients. To investigate the stability of an equilibrium state under rigid loading we must study the second variation of $U(W_i)$ with the restraint $\Delta\epsilon = 0$. Clearly however we are free to study the second variation of the dead-load energy function

$$V(P, W_i) = U(W_i) - P\epsilon(W_i)$$

with the same restraint.

Thus while the external stability of an equilibrium state is determined by the second variation of V given by

$$\delta^2 V = \frac{1}{2} \sum \frac{\partial^2 V}{\partial u_i^2} u_i^2 \equiv \frac{1}{2} \sum C_i u_i^2$$

the internal stability will be determined by the same expression with the constraint

$$\psi = \delta\epsilon = \sum S_i u_i = 0$$

The acceptance of this statement without qualification is the essentially 'linear' and non-rigorous feature of the following analysis.

Two results are immediately seen. If the structure is stable under dead loading, so that all the C_i are positive, the structure must be stable under rigid loading. Secondly, if the structure has two or more degrees of instability under dead load (so that two or more of the C_i are negative) then it is unstable under rigid loading: this conclusion follows immediately from a result of Courant and Hilbert (1953). Thus the only case of further interest arises when a single dead-load stability coefficient is negative.

Let us write

$$Z \equiv \delta^2 V = \frac{1}{2} \sum C_i u_i^2$$

and determine the $m - 1$ stationary values of Z on the 'sphere'

$$\sum u_i^2 = 2$$

with the constraint

$$\psi = \sum S_i u_i = 0$$

Without the constraint, the m stationary points of Z occur on the u_i axes, to give the dead-load stability coefficients $Z = C_1$, $Z = C_2$, ... $Z = C_m$. With the constraint, the stationary values will represent a set of rigid-load stability coefficients.

Introducing two Lagrange multipliers $[(1/2)\rho_1]$ and ρ_2 we consider the auxiliary problem of locating the stationary points of

$$\overline{Z} = \frac{1}{2} \sum C_i u_i^2 - \frac{1}{2} \rho_1 \left\{ \sum u_i^2 - 2 \right\} - \rho_2 \sum S_i u_i$$

The subsequent analysis is presented in detail in the Appendix, and the required stationary values of Z are given by the $m - 1$ values of ρ_1 satisfying the equation

$$\underline{\sum \frac{S_i^2}{C_i - \rho_1} \Bigg/ \sqrt{\sum \frac{S_i^2}{(C_i - \rho_1)^2}} = 0} \qquad \text{———} \tag{15a}$$

which can in general be simplified by omitting the denominator to give

$$\underline{\sum \frac{S_i^2}{C_i - \rho_1} = 0} \qquad \text{———} \tag{15b}$$

Thus a set of stability coefficients for rigid loading is given by the $m - 1$ roots of equation (15).

Moreover for a change of internal stability this equation must be satisfied by $\rho_1 = 0$. It follows that the general condition for a change of internal stability can be written as

$$\underline{\sum (S_i^2/C_i) = 0} \qquad \text{———} \tag{16}$$

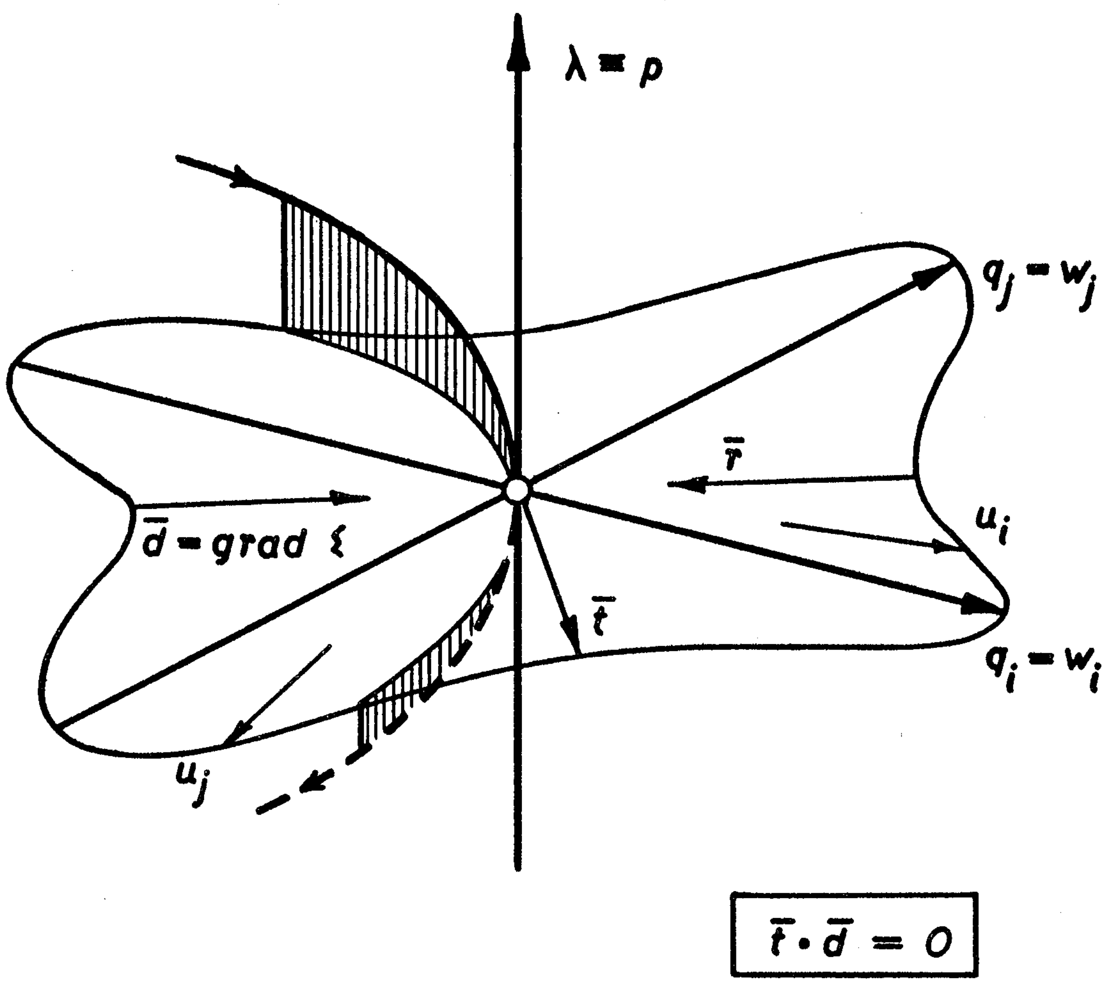


FIG. 8. RIGID-LOAD SNAPPING CONFIGURATION IN LOAD-COORDINATE SPACE

6.4. Snapping Condition. In general a loss of internal stability will not coincide with the vanishing of any of the S_i or C_i coefficients. Moreover, when all the C_i coefficients are non-zero, a loss of internal stability will be characterized by the vanishing of

$$\sum (S_i^2/C_i)$$

as indicated by equation (16).

Now we have already established that

$$\left|\frac{\partial \epsilon}{\partial P}\right|_{\text{path}} = \sum (S_i^2/C_i) \qquad \text{(equation 13)}$$

It follows that the initial internal stability of a structure will in general be lost at a snapping point, at which the equilibrium path of the structure reaches the first locally maximum value of ϵ.

The previous discussion of a rigid-load snapping point is thus confirmed.

A general critical path configuration is shown in Figure 8. It should be noted that the equilibrium path yields a maximum value of ϵ when the response vector $\vec{t}$ has no component in the direction of the disturbance vector $\vec{d} = \overrightarrow{\text{grad } \epsilon}$.

It has been indicated that an equilibrium state of a structure cannot be unstable under rigid loading if it is stable under dead loading. Thus a rigid-load snapping point will in general be encountered after a dead-load snapping point, as illustrated by the equilibrium paths of Figure 3.

6.5 Buckling Condition. If one of the S_i coefficients drops to zero, equation (15b) loses one of its roots, and the missing solution will be supplied by the complete equation, (15a). The necessary analysis is presented in the Appendix, and it is shown that if $S_1 = 0$, the missing solution is given by $\rho_1 = C_1$, if all the C_i are distinct. That is to say, if one of the S_i coefficients is equal to zero, equation (15b) will supply $m - 2$ internal stability coefficients, and the missing stability coefficient will be equal to the corresponding dead-load stability coefficient.

It is clear that this special solution corresponds to the fact that with $S_1 = 0$, the expression

$$\sum S_i u_i = 0$$

offers no constraint in the principal direction $\vec{j}_1$.

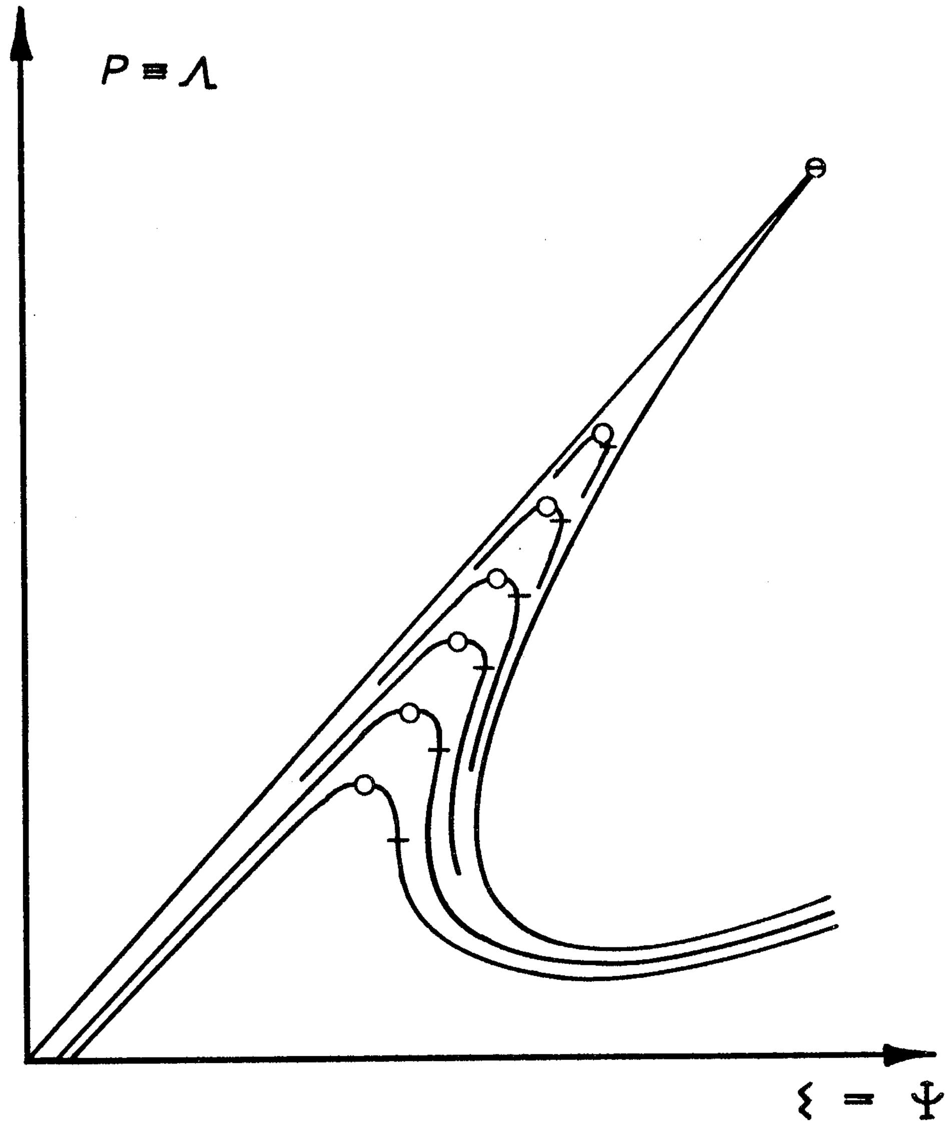
Since with $S_1 = 0$ one internal stability coefficient is given by $\rho_1 = C_1$, it follows that, as a special case, the internal stability of a structure will be 'lost' if S_1 and C_1 vanish simultaneously. In this case the first term of equation (16) is indeterminate, being essentially $0^2/0 \cdot 0$.

As we have seen in section 4, the simultaneous vanishing of S_1 and C_1 yields a point of intersection, so the previous discussion of rigid-load buckling is confirmed.

The above discussion demonstrates that the simultaneous vanishing of S_1 and C_1 as defined in a direct dead-load analysis implies the vanishing of C_1 as defined in a direct rigid-load analysis. The simultaneous vanishing of the dead-load coefficients must also imply the vanishing of S_1 in a direct rigid-load analysis, and a proof of this inter-relationship is readily derived from the equality

$$\frac{\partial^2 V}{\partial \epsilon \partial u_1} = \frac{\partial^2 U}{\partial \epsilon \partial u_1}$$

It is instructive to note that a general branching point can be regarded as the coincidence of two snapping points; that is the coincidence of an extremum with respect to P, and an extremum with respect to ϵ. Thus as we approach a dead-load snapping point we observe C_1 dropping to zero, while as we approach a rigid-load snapping point we observe the expression



○ $c_1 = 0$

– $\sum s_i^2/c_i \,/\!/\, \sum s_i^2/c_i^2 = 0$

⊖ $c_1 = s_1 = 0$

FIG. 9. BUCKLING POINT VIEWED AS THE COINCIDENCE OF TWO SNAPPING POINTS

$$\sum (S_i^2/C_i) \Big/ \sqrt{\sum (S_i^2/C_i^2)}$$

tending to zero. These two conditions can only be realized simultaneously if S_1 is also dropping to zero; that is if we are approaching a point of intersection with $S_1 = C_1 = 0$.

This phenomenon is clearly visible in Figure 9, which might represent the behaviour of a constrained Euler strut under axial compression, with varying degrees of initial imperfection (Tsien, 1942). As the strut under consideration slowly changes with decreasing inperfection, the two snapping points merge to give the branching point of the perfect strut.

6.6. Internal Stability With External Instability. An equilibrium state of a structure might be internally and externally stable, or internally and externally unstable. The only other possibility, since internal instability implies external instability, is that the state is internally stable but externally unstable.

It is of some interest to examine the conditions under which this last possibility can be realized.

Restricting attention to a perfectly general non-critical equilibrium state, let us suppose that $\bar{\rho}_1$, the smallest root of the equation

$$\sum (S_i^2)/(C_i - \rho_1) = 0$$

is positive, so that the basic state is internally stable. Then if we suppose that $C_1 < C_2 < C_3 \ldots < C_m$, it follows from a result of Courant and Hilbert (1953) that $C_1 < \bar{\rho}_1 < C_2$. Thus C_s must be positive for s not equal to one, while C_1 may be positive or negative.

Let us suppose that C_1 is negative, and consider the two expansions

$$\sum \frac{S_i^2}{C_i - \bar{\rho}_1} \equiv \frac{S_1^2}{C_1 - \bar{\rho}_1} + \frac{S_2^2}{C_2 - \bar{\rho}_1} + \ldots + \frac{S_m^2}{C_m - \bar{\rho}_1} = 0$$

$$\left|\frac{\partial \epsilon}{\partial P}\right|_{\text{path}} = \sum \frac{S_i^2}{C_i} \equiv \frac{S_1^2}{C_1} + \frac{S_2^2}{C_2} + \ldots + \frac{S_m^2}{C_m}$$

We see that in each expansion the first term is negative, and the remaining terms are positive. Moreover in changing from the first line to the second line, the negative term has increased in absolute magnitude, while the following positive terms have all decreased in magnitude. Clearly the sum of the second series must be negative.

Thus if $\bar{\rho}_1$ is positive and C_1 is negative, the slope

$$\left|\frac{\partial \epsilon}{\partial P}\right|_{\text{path}}$$

must be negative. Moreover if C_1 is positive it is clear that the slope will be positive.

It follows that if an equilibrium state of a structure is internally stable it will also be externally stable unless the equilibrium path passing through that state has a negative slope on a plot of the load against the corresponding deflection. That is to say a state of internal stability and external instability will always be associated with a negative slope on a plot of P against ϵ.

A rigorous proof of this result has been presented in an earlier paper (Thompson 1961a), in which it was shown to be of considerable value in establishing the external stability of certain post-snapping equilibrium states.

VII. RESUME

The stability of an elastic structure subjected to a single generalized load is discussed. Dead and rigid loading conditions, and the associated external and internal stabilities of the structure are considered.

Two sets of generalized coordinates are introduced to define the deformations of the structure. A set of m coordinates is used to describe the deformed state of the structure when, under dead loading conditions, the corresponding displacement is unconstrained. A set of $m - 1$ coordinates is used to describe the deformed state of the structure when, under rigid loading conditions, values of the corresponding displacement are prescribed.

The stability of a structural system described by an energy function, $V(Q_i, \Lambda)$, has recently been discussed by the author: here the Q_i are a set of n generalized coordinates, and Λ is a 'loading parameter'. Clearly if n is set equal to m, and the loading parameter Λ is equated to the magnitude of the load, this general theory is directly applicable to the dead-load problem. Conversely if n is set equal to $m - 1$, and the loading parameter is equated to the magnitude of the imposed displacement, the general theory is directly applicable to the rigid-load problem.

Thus the general theory, suitably interpreted, can serve as a discussion of the internal or external stability of a structure.

The presentation can moreover be further unified by the remarkable reciprocal properties of the 'auxiliary loading parameter' Ψ, which we define by the equation $\Psi = -(\partial V/\partial \Lambda)$. It is seen that when the loading parameter Λ is equated to the magnitude of the generalized load in a discussion of dead loading, the auxiliary parameter Ψ can be identified as the corresponding deflection of the load. Conversely, when the loading parameter Λ is equated to the magnitude of the imposed displacement, the auxiliary loading parameter Ψ can be identified, with a change of sign, as the magnitude of the constraining load.

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Keeping in mind the dual interpretation, the general theory can be summarized as follows.

The energy function $V(Q_i, \Lambda)$ is first expanded as a power or Taylor series about an equilibrium state of interest, lower case symbols being introduced to denote changes in the variables from this state. Then, introducing a locally principal set of coordinates u_i, we can write to a first approximation

$$v = \frac{1}{2} \sum C_i u_i^2 - \lambda \sum S_i u_i$$

if we agree to ignore certain inactive terms which do not involve the coordinates. Here the C_i and S_i coefficients are constants for a given basic equilibrium state, the C_i representing a set of stability coefficients that defines the stability of the basic state.

The equilibrium path passing through the basic state is defined by the n equations $(\partial v/\partial u_i) = 0$, which can in general be solved to give

$$u_r = \lambda \frac{S_r}{C_r} \qquad \text{for all } r$$

Further, since local changes in Ψ are given to a first approximation by

$$\psi = \sum S_i u_i + T\lambda$$

where T is a constant, we can in general write

$$\left|\frac{\partial \Psi}{\partial \Lambda}\right|_{\text{path}} = \sum \frac{S_i^2}{C_i} + T$$

The stability of the basic equilibrium state is determined by the stability coefficients, C_i, which vary in a continuous manner as the basic state under consideration is allowed to move along an equilibrium path. It follows that the initial stability of a structure can only be lost at a critical equilibrium state for which one of these coefficients, C_1 say, has dropped to zero.

If S_1 is not equal to zero, the vanishing of C_1 implies the vanishing of λ/u_1 in the linearized analysis. A non-linear analysis shows that under these conditions the equilibrium path will in general yield a locally maximum or minimum value of Λ at the critical equilibrium state. The path thus exhibits the well-known snapping configuration.

If however S_1 vanishes simultaneously with C_1, we see that λ/u_1 is then linearly indeterminate. In this case it can be shown that in general two equilibrium paths intersect at the critical equilibrium state to give a point of bifurcation.

Thus we see that under either dead or rigid loading conditions, the initial stability of a structure will in general be lost at a snapping point $(C_1 = 0)$, at which the equilibrium path of the structure reaches the first locally maximum value of the imposed loading parameter. As a special case $(C_1 = S_1 = 0)$, the initial internal and external stabilities of the structure may be lost simultaneously at a point of intersection.

The two direct treatments inherent in this unified presentation are essentially distinct when applied to the same structural problem. For this reason it is instructive to study the internal stability of a structure within the framework of a dead-load analysis, as follows.

Consider the dead-load analysis supplied by equating Λ to the magnitude of the generalized load, in which Ψ can be identified as the corresponding deflection. The C_i and u_i are now a set of m dead-load stability coefficients and principal coordinates respectively. The constant $T \equiv -(\partial^2 V/\partial\Lambda^2)$ is identically zero, and changes in the corresponding deflection are given to a first approximation by

$$\psi = \sum S_i u_i$$

It follows that the internal stability of the structure is in general determined by the quadratic form

$$Z = \frac{1}{2} \sum C_i u_i^2$$

with the constraint

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$$\psi = \sum S_i u_i = 0$$

Thus the stationary values of Z on the 'sphere'

$$\sum u_i^2 = 2$$

with the above constraint will supply a set of internal stability coefficients, which are given in general by the $m - 1$ roots of the equation

$$\sum \frac{S_i^2}{C_i - Z} = 0$$

For a loss of stability this equation must be satisfied by $Z = 0$, which gives the general condition for a loss of internal stability as

$$\sum \frac{S_i^2}{C_i} = 0$$

Thus since

$$\left|\frac{\partial \Psi}{\partial \Lambda}\right|_{path} = \sum \frac{S_i^2}{C_i}$$

we see that a loss of internal stability will in general be associated with an infinite slope on a plot of the load against the corresponding deflection. The discussion of rigid-load snapping inherent in the general analysis is thus confirmed.

The general equation supplying the internal stability coefficients loses one of its roots if S_1 is equal to zero, and a more careful study of the problem shows that the missing solution is given by $Z = C_1$. That is to say if $S_1 = 0$, one of the internal stability coefficients is equal to the corresponding external stability coefficient, C_1. Clearly as a special case not covered by the general criterion, the internal stability of a structure will be lost if S_1 and C_1 vanish simultaneously.

As we have seen earlier the simultaneous vanishing of S_1 and C_1 in general yields a point of intersection. The previous conclusion that a point of intersection will be associated with a loss of internal stability is thus confirmed.

It is finally shown that if an equilibrium state of a structure is internally stable it will also be externally stable unless the equilibrium path passing through that state has a negative slope on a plot of the load against the corresponding deflection.

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APPENDIX I

The evaluation of the stationary values of

$$Z = \frac{1}{2} \sum C_i u_i^2$$

on the 'sphere'

$$\sum u_i^2 = 2$$

with the constraint

$$\sum S_i u_i = 0$$

is here presented in detail. In this way the range of validity of the 'general' results quoted in section 6 is made apparent.

To determine the stationary values of Z we introduce the two Lagrange multipliers $(1/2)\rho_1$ and ρ_2, and consider the auxiliary function

$$\bar{Z} = \frac{1}{2} \sum C_i u_i^2 - \frac{1}{2} \rho_1 \left(\sum u_i^2 - 2 \right) - \rho_2 \sum S_i u_i$$

The stationary points of this function are also the stationary points of Z, so for the stationary values we must evaluate Z under the conditions

$$\underline{\sum u_i^2 = 2} \tag{A}$$

$$\underline{\sum S_i u_i = 0} \tag{B}$$

and

$$\underline{\frac{\partial \bar{Z}}{\partial u_r} = C_r u_r - \rho_1 u_r - \rho_2 S_r = 0} \qquad \text{for all } r \tag{C}$$

We can observe that these conditions represent $n + 2$ equations for the $n + 2$ 'unknowns' u_i, ρ_1 and ρ_2.

If we multiply the r^{th} equation of C by u_r, repeating this procedure for all values of r from 1 to n, and then sum the resulting n equations, we have

$$\sum C_i u_i^2 - \rho_1 \sum u_i^2 - \rho_2 \sum S_i u_i = 0$$

Now using equations A and B this gives

$$Z = \frac{1}{2} \sum C_i u_i^2 = \rho_1$$

Thus the evaluation of the stationary values of Z reduces to the solution of equations (A), (B) and (C) for ρ_1.

Equation (C) gives

$$u_r = \frac{\rho_2 S_r}{C_r - \rho_1} \qquad \text{for all } r$$

Substituting this expression for u_r in equation (A) gives

$$\underline{\rho_2^2 \sum \frac{S_i^2}{(C_i - \rho_1)^2} = 2} \tag{D}$$

and substituting the expression for u_r in equation (B) gives

$$\rho_2 \sum \frac{S_i^2}{C_i - \rho_1} = 0 \tag{E}$$

Finally eliminating ρ_2 (which might of course be zero) from these two equations we have the single equation for ρ_1,

$$\sum \frac{S_i^2}{C_i - \rho_1} \Bigg/ \sqrt{\sum \frac{S_i^2}{(C_i - \rho_1)^2}} = 0 \tag{F}$$

The $n - 1$ roots of this equation will supply the required stationary values of Z. We shall write $\rho_1 = \bar{\rho}_1$ to represent a particular solution of the equation.

Considering the evaluation of a particular solution, and with a view to dropping the denominator from the left-hand side, let us examine under what conditions the denominator of equation F can become infinite.

We shall assume that all the C_i coefficients are distinct, so that $C_i \neq C_j$ for all $i \neq j$. Clearly then the denominator can only tend to infinity if $\bar{\rho}_1$ tends to one of the C_i coefficients, C_1 say. Moreover S_1 must be of an order greater than $C_1 - \bar{\rho}_1$, so that $(C_1 - \bar{\rho}_1)/S_1$ is tending to zero.

Assuming then that $\bar{\rho}_1 \to C_1$ and $S_1 \gg C_1 - \bar{\rho}_1$, only the first term of the denominator need be retained, and equation (F) implies that

$$\frac{C_1 - \bar{\rho}_1}{S_1} \left(\frac{S_1^2}{C_1 - \bar{\rho}_1} + \frac{S_2^2}{C_2 - \bar{\rho}_1} + \ . \ . \ . \right) = 0$$

Considering now the first term in the brackets, we see that this equation implies that S_1 must be tending to zero.

Thus the only solutions that the denominator can supply are represented by $\bar{\rho}_1 = C_r$, which arises when S_r is equal to zero.

It follows that if no S_i coefficient is equal to zero, and if all the C_i coefficients are distinct, we can omit the denominator and write

$$\underline{\sum \frac{S_i^2}{C_i - \rho_1} = 0} \qquad \text{———————} \qquad (G)$$

The roots of this equation will in general yield the $n - 1$ stationary values of Z.

If however one of the S_i coefficients drops to zero, this equation loses one of its roots, and we have seen that the missing solution is supplied by the denominator of equation F. More specifically, if p of the S_i, S_1, S_2, ... S_p, are equal to zero, equation (G) will supply $n - 1 - p$ stationary values of Z, and the missing solutions will be given by $\rho_1 = C_1$, $\rho_1 = C_2$, ... $\rho_1 = C_p$, provided that all the C_i are distinct.

If one of the stationary values of Z is to be equal to zero, equation (F) must be satisfied by $\rho_1 = 0$. Thus the condition for a vanishing stationary value can be written as

$$\underline{\sum \frac{S_i^2}{C_i} \Bigg/ \sqrt{\sum \frac{S_i^2}{C_i^2}} = 0} \qquad \text{(H)}$$

If moreover no C_i is equal to zero, the denominator can be omitted, so that we have

$$\underline{\sum \frac{S_i^2}{C_i} = 0} \qquad \text{(I)}$$

which is the 'general' condition for a vanishing stationary value of Z.

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APPENDIX II

The behaviour of a simple two-degree-of-freedom buckling model is here used to illustrate the salient features of the paper, attention being restricted to the initial equilibrium path of the model. The loading of the model can be either dead or rigid in nature, and the initial internal and external stabilities of the system are lost simultaneously at a general point of bifurcation.

The behaviour of the model under dead load is studied, along the lines developed in the paper, for two initial sets of coordinates.

The first dead-load analysis involves a straight-forward choice of the initial coordinates. These coordinates are always principal along the 'unbuckled' path, and the disturbance vector is always directed along the 'first' principal axis. The point of bifurcation thus arises when the 'second' stability coefficient drops to zero.

A more instructive illustration of the buckling phenomenon is obtained by studying the same problem in a more-general set of coordinates. The analysis is thus repeated with a new set of initial coordinates, a subsequent transformation of coordinates now being necessary, since the new initial coordinates are not principal along the equilibrium path. The principal axes and the disturbance vector are seen to rotate as the basic state moves along the equilibrium path, and the point of bifurcation arises when a stability coefficient and the corresponding component of the disturbance vector vanish simultaneously.

Under rigid load the model has effectively one degree of freedom, and its behaviour is studied, following the lines of the paper, for two choices of the initial coordinate. In the first analysis the disturbance vector is identically zero along the equilibrium path, while in the second analysis the vector is not identically zero, but vanishes with the stability coefficient at the critical equilibrium state.

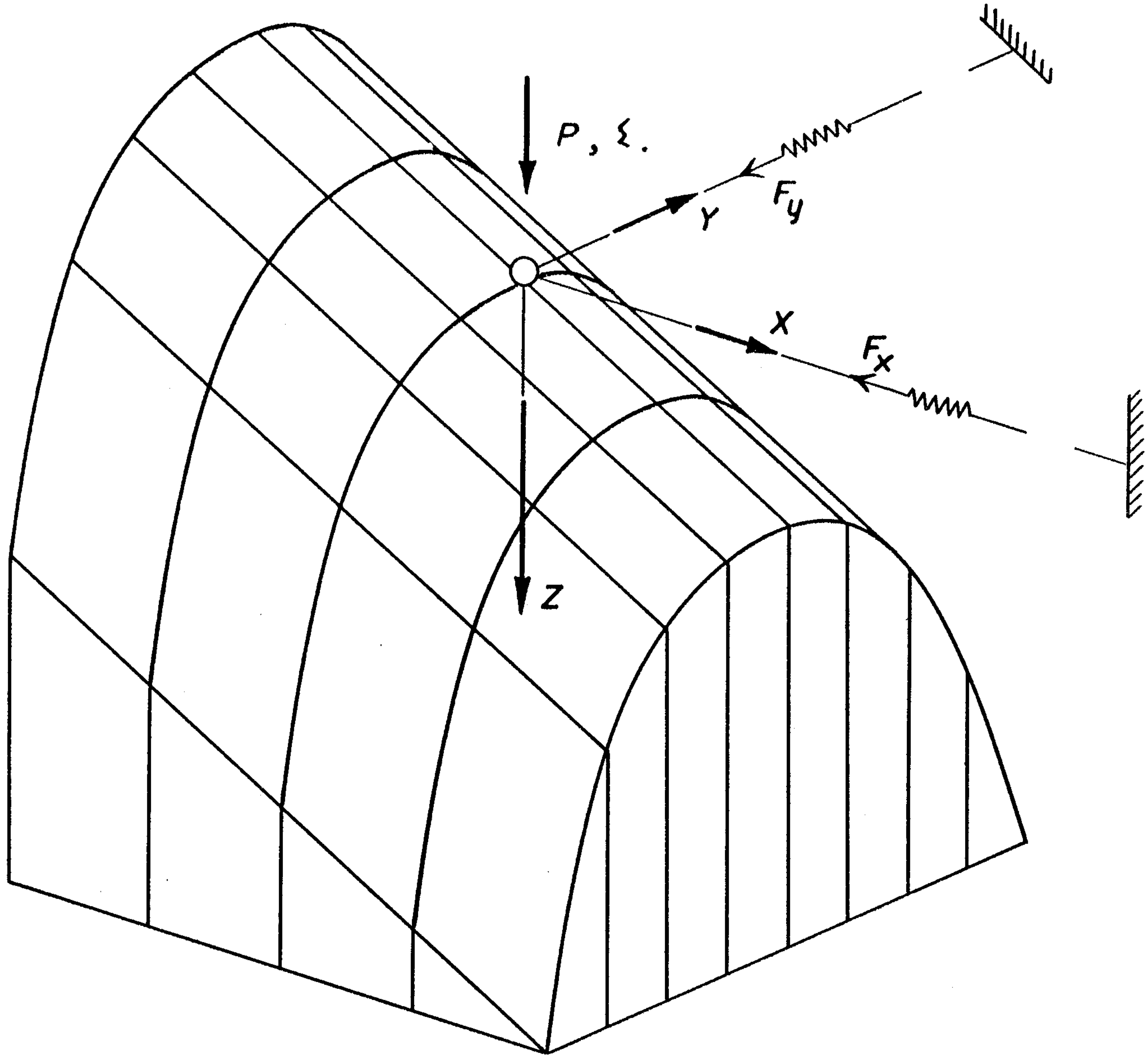


FIG. 10. A TWO-DEGREE-OF-FREEDOM BUCKLING MODEL

1. Structural System

The buckling model, illustrated in Figure 10, consists of a smooth particle resting on the surface $Z = X + \frac{1}{2} Y^2$ constrained by two springs lying in the horizontal XY plane, and loaded by a vertical force P. The 'X' spring is linear, with

$$\underline{F_x = X} \quad \text{and} \quad \underline{U_x = \frac{1}{2} X^2},$$

while the 'Y' spring is non-linear, with

$$\underline{F_y = Y - \frac{1}{2} Y^2} \quad \text{and} \quad \underline{U_y = \frac{1}{2} Y^2 - \frac{1}{6} Y^3}$$

for $Y < 3$. Here F_x and F_y, and U_x and U_y are the constraining forces and the strain energies of the two springs.

When the loading is dead, it follows that the total potential energy of the system can be written as

$$\underline{V = \frac{1}{2} X^2 + \frac{1}{2} Y^2 - \frac{1}{6} Y^3} \qquad \text{(A)}$$
$$\underline{-P \left\{ X + \frac{1}{2} Y^2 \right\}}$$

2. Dead-Load Analysis in the Basic Coordinate System

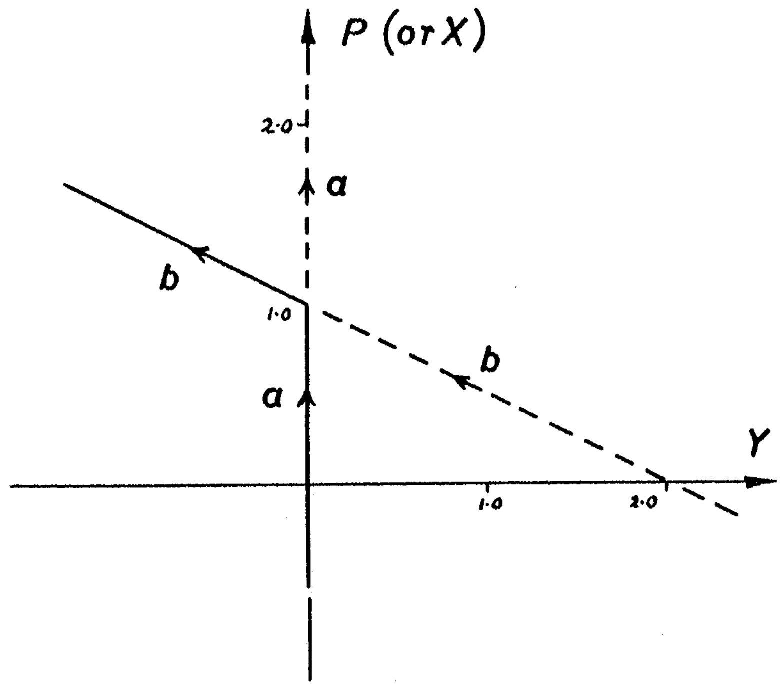
Let us first consider the dead-load problem using the basic (X,Y) coordinate system.

2.1 Equilibrium Paths. The total potential energy is given by equation (A), and setting

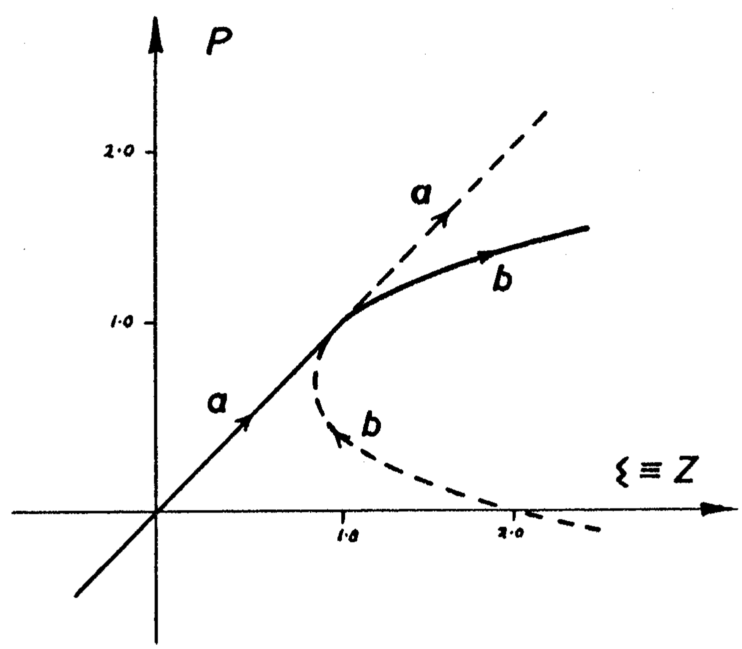
$$\underline{\frac{\partial V}{\partial X} = 0} \qquad \text{gives} \qquad \underline{X = P} \;;$$

setting $$\underline{\frac{\partial V}{\partial Y} = 0} \qquad \text{gives} \qquad \underline{Y = 0}$$

$$\text{or} \qquad \underline{Y = 2(1-P)}$$



a) LOAD AGAINST THE LATERAL DEFLECTION



b) LOAD AGAINST THE CORRESPONDING DEFLECTION

FIG. 11. EQUILIBRIUM PATHS IN THE BASIC COORDINATE SYSTEM

The corresponding deflection, $\epsilon \equiv Z$, is thus given by

$$\underline{\epsilon = P} \qquad \text{or} \qquad \underline{\epsilon = P + 2(1-P)^2} \quad .$$

Clearly, we have two equilibrium paths exhibiting an exchange of stabilities at the 'general' point of bifurcation,

$$\underline{P = X = 1, \quad Y = 0} \; .$$

The equilibrium paths are shown in Figure 11, in which a broken curve indicates a region of external instability.

2.2 Loaded Equilibrium State - Following the general discussions of the paper, let us consider the loaded 'unbuckled' equilibrium state,

$$\underline{P = P_o \, , \quad X = X_o = P_o \, , \quad Y = 0 \, ,} \tag{B}$$

and let us write

$$\underline{p = P-P_o} \, , \quad \underline{x = X-X_o} \, , \quad \underline{y = Y} \; .$$

The total potential energy can now be written as

$$\underline{V = -\tfrac{1}{2}P_o^2 + \tfrac{1}{2}x^2 + (1-P_o)\tfrac{1}{2}y^2 - (1/6)y^3} \\ \underline{-p\left\{P_o + x + \tfrac{1}{2}y^2\right\}} \tag{C}$$

We see that along the initial path the (x,y) coordinate system is always principal. The stability coefficients are

$$\underline{C_x \equiv \frac{\partial^2 V}{\partial X^2} = 1} \\ \underline{C_y \equiv \frac{\partial^2 V}{\partial Y^2} = 1 - P_o} \tag{D}$$

and the components of the disturbance vector are

$$\underline{S_x \equiv \frac{\partial Z}{\partial X} \equiv -\frac{\partial^2 V}{\partial P \partial X} = 1} \\ \underline{S_y \equiv \frac{\partial Z}{\partial Y} \equiv -\frac{\partial^2 V}{\partial P \partial Y} = 0} \tag{E}$$

Viewed in the basic coordinates (X,Y), the behaviour of the system in the vicinity of the initial path is thus extremely simple. The (X,Y) coordinates themselves are always principal, the component S_y is always zero, and the stability coefficient C_y drops to zero at the critical equilibrium state.

3. Dead-load Analysis in a More-general Coordinate System

It is instructive to study the dead-load problem in a more-general coordinate system. In this way it will be seen that the simple behaviour observed above is a property of the coordinates employed, rather than a property of the system itself.

3.1 Definition of Coordinates - We define, somewhat arbitrarily a new initial set of coordinates by the equations

$$\begin{aligned} Q_1 &= \tfrac{1}{2}(X-Y) \\ Q_2 &= \tfrac{1}{2}(X+Y) - \tfrac{1}{4}(X-Y)^2 \end{aligned} \tag{F}$$

These equations can be inverted to give

$$\begin{aligned} X &= Q_1 + Q_2 + Q_1^2 \\ Y &= Q_2 - Q_1 + Q_1^2 \end{aligned} \tag{G}$$

and we can write the Jacobian determinant of the transformation as

$$\begin{vmatrix} \dfrac{\partial X}{\partial Q_1} & \dfrac{\partial X}{\partial Q_2} \\[2ex] \dfrac{\partial Y}{\partial Q_1} & \dfrac{\partial Y}{\partial Q_2} \end{vmatrix} = 2$$

Curves of constant X and constant Y are shown on a plot of Q_1 against Q_2 in Figure 12, and it can be observed that the two sets of coordinates are not mutually orthogonal.

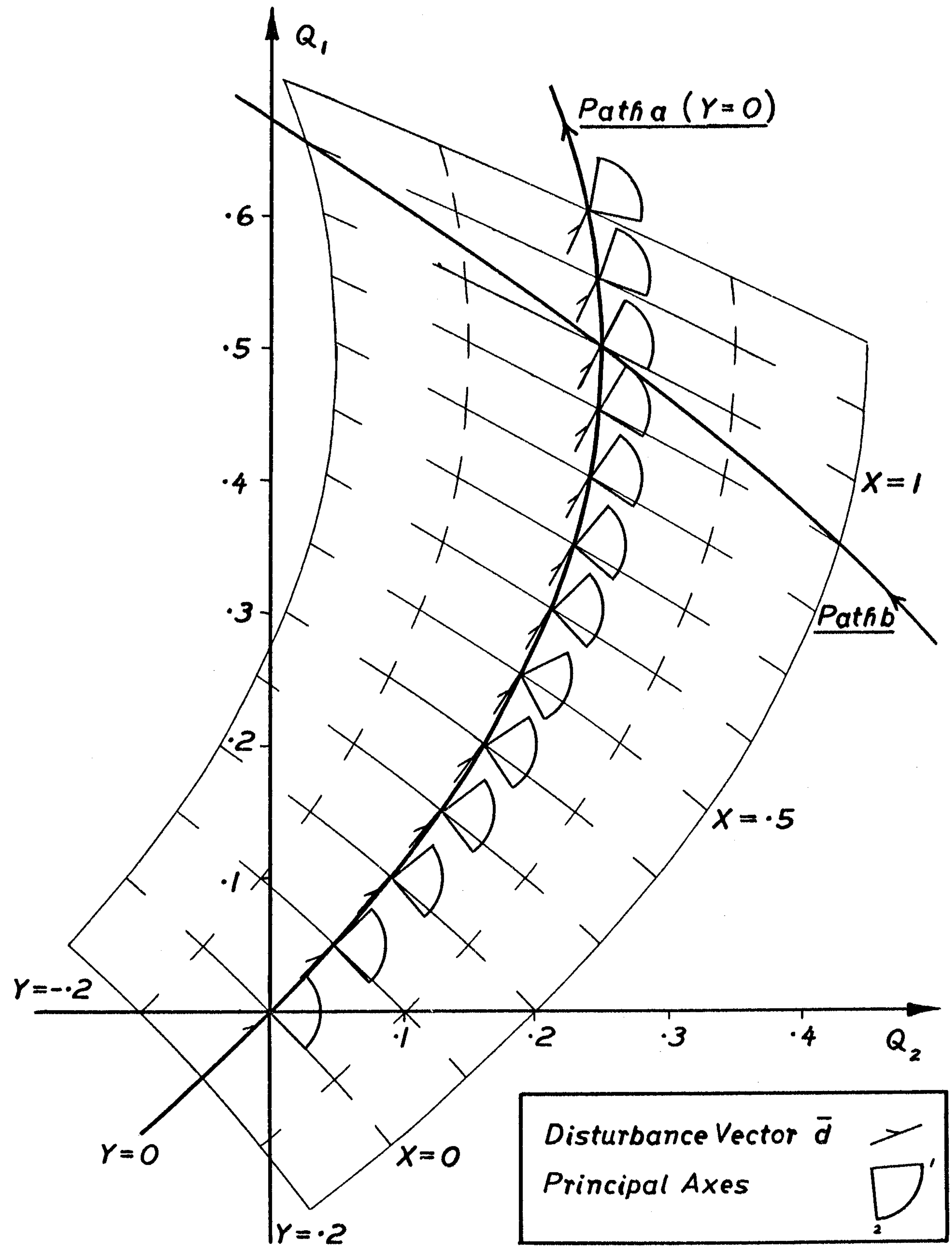


FIG. 12. EQUILIBRIUM PATHS IN THE GENERALIZED COORDINATE SYSTEM

3.2 Loaded Equilibrium State - Considering changes from the basic equilibrium state of equation (B), linearizing the energy expansion, and dropping certain inactive terms, the change in energy from the basic state can be written as

$$\underline{\nu = \tfrac{1}{2} q_1^2 \left(2 - P_o + 4 P_o^2 - P_o^3 \right) + \tfrac{1}{2} q_2^2 \left(2 - P_o\right)}$$
$$\underline{+ q_1 q_2 \left(3 P_o - P_o^2\right) - p \left\{ (1 + P_o)\, q_1 + q_2 \right\}} \qquad \text{(H)}$$

We observe that the q_i coordinates are not principal.

Following the lines of the general discussion we must now find the orthogonal transformation that will diagonalize the quadratic form of ν. Any orthogonal transformation of two variables can be written in the form

$$\underline{u_1 = q_1 \cos \alpha - q_2 \sin \alpha}$$
$$\underline{u_2 = q_1 \sin \alpha + q_2 \cos \alpha} \qquad \text{(I)}$$

so we shall use these equations to define the new coordinates u_i, choosing α to eliminate the quadratic cross-term of ν.

The necessary value of α is given by the equation

$$\underline{\operatorname{Tan} 2\alpha = \frac{2 P_o - 6}{4 P_o - P_o^2}}$$

and in the principal coordinates u_i we have finally

$$\underline{\nu = \tfrac{1}{2} C_1 u_1^2 + \tfrac{1}{2} C_2 u_2^2}$$
$$\underline{- p \left\{ S_1 u_1 + S_2 u_2 \right\}}$$

where

$$\underline{C_1 = \tfrac{1}{2} \left(J + P_o H \right)}$$
$$\underline{C_2 = \tfrac{1}{2} \left(J - P_o H \right)} \qquad \text{(J)}$$

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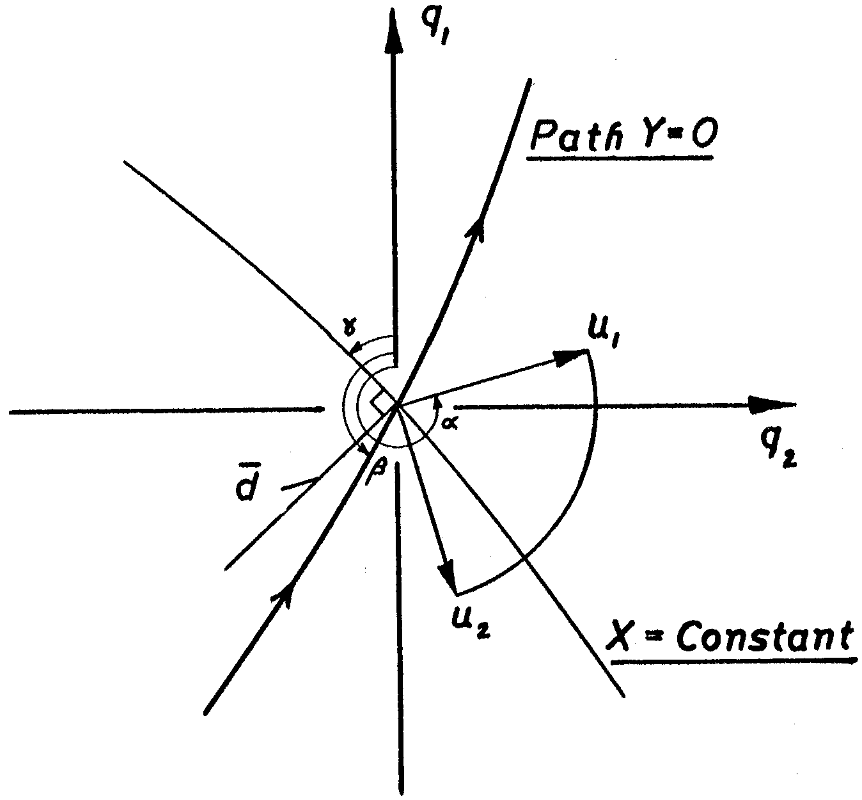

FIG. 13. BASIC EQUILIBRIUM STATE OF THE GENERALIZED ANALYSIS

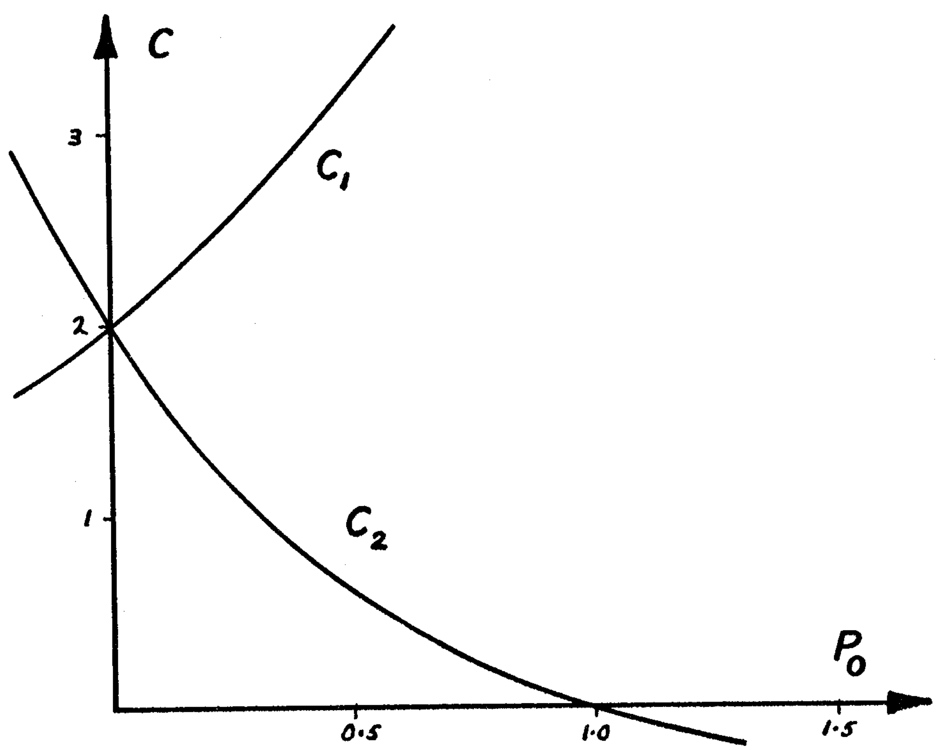

FIG. 14. STABILITY COEFFICIENTS OF THE GENERALIZED ANALYSIS

and $$S_1 = \cos\alpha - \sin\alpha + P_o \cos\alpha$$

$$S_2 = \cos\alpha + \sin\alpha + P_o \sin\alpha .$$

Here $$J \equiv 4 - 2P_o + 4P_o^2 - P_o^3$$

and $$H \equiv + \sqrt{36 - 24 P_o + 20 P_o^2 - 8 P_o^3 + P_o^4} .$$

The basic equilibrium state being defined by the value of P_o, it is seen that the principal axes rotate as the basic state is allowed to move along the equilibrium path. Moreover, no energy coefficient is identically zero along the path, and it is readily verified that the two coefficients C_2 and S_2 vanish simultaneously at the critical equilibrium state for which $P_o = 1$.

The rotation of the principal axes (slightly exaggerated for clarity), and the rotation of the disturbance vector are shown in Fig. 12. The relevant angles are defined in Fig. 13, and can be evaluated from the equations

$$\operatorname{Tan} 2\alpha = -\frac{2(3-P_o)}{P_o(4-P_o)}$$

$$\operatorname{Tan} 2\beta = -\frac{2(1-P_o)}{P_o(2-P_o)}$$

$$\operatorname{Tan} 2\gamma = -\frac{2(1+P_o)}{P_o(2+P_o)}$$

The variations of C_1 and C_2 with P_o are shown in Figure 14.

4. Rigid - load Analysis in the Basic Coordinate System

Let us now study the initial equilibrium path of the model in the context of a rigid-load analysis following the lines of the general analysis of the paper.

When the corresponding deflection, ϵ, is constrained the model has one degree of freedom, and we shall here choose Y as the single coordinate. The appropriate energy function is the sum of the strain energies U_x and U_y, so we can write

$$\underline{V \equiv U_x + U_y = \left\{\tfrac{1}{2}Y^2 - (1/6)Y^3 + (1/8)Y^4\right\}}$$
$$\underline{- \epsilon \left\{\tfrac{1}{2}Y^2\right\} + \tfrac{1}{2}\epsilon^2}$$

In contrast to the energy function of the dead-load analysis, we see that V is not linear in the loading parameter $(\epsilon \equiv \Lambda)$.

Considering changes from the basic state of equation (B), we have

$$\nu = \underline{\left\{\tfrac{1}{2}y^2(1-\epsilon_o) - (1/6)y^3 + (1/8)y^4\right\}} \qquad \text{(K)}$$
$$\underline{- e\left\{-\epsilon_o + \tfrac{1}{2}y^2\right\} + \tfrac{1}{2}e^2}$$

where e is the change in ϵ. Thus the single rigid-load stability coefficient C_y and the single component of the disturbance vector S_y are given by

$$\underline{C_y \equiv \frac{\partial^2 V}{\partial Y^2} = 1 - \epsilon_o = 1 - P_o}$$

$$\underline{S_y \equiv -\frac{\partial^2 V}{\partial\epsilon\partial Y} = 0}$$

We see that along the path the disturbance vector is identically zero, and the stability coefficient drops to zero at the critical equilibrium state for which $P_o = 1$.

5. <u>Rigid-load Analysis in a More-general Coordinate System</u>

It is instructive to repeat the rigid-load analysis using a 'more-general' coordinate, Q, which we define as follows

$$\underline{Q = X - Y + \tfrac{1}{2}Y^2}$$

The appropriate energy function is now

$$V \equiv U_x + U_y = \left\{ \tfrac{1}{2}Q^2 + (1/6)Q^3 + (1/8)Q^4 \right\}$$
$$- \epsilon \left\{ Q + Q^2 + \tfrac{1}{2} Q^3 \right\}$$
$$+ \epsilon^2 \left\{ 1 + \frac{3}{2} Q + \frac{3}{4} Q^2 \right\}$$
$$- \epsilon^3 \left\{ \frac{2}{3} + \frac{1}{2} Q \right\}$$
$$+ \epsilon^4 \left\{ \frac{1}{8} \right\}$$

so that

$$C_Q \equiv \frac{\partial^2 V}{\partial Q^2} = 1 - \epsilon_o = 1 - P_o$$

$$S_Q \equiv \frac{\partial^2 V}{\partial \epsilon \partial Q} = 1 - \epsilon_o = 1 - P_o$$

We see that the disturbance vector is not now identically zero along the path, but vanishes with the stability coefficient at the critical equilibrium state for which $P_o = 1$.

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